A KEY

TO THE

LAST NEW-YORK EDITION

OF

BONNYCASTLE'S ALGEBRA;

AND ALSO ADAPTED TO THE FORMER AMERICAN AND LATEST LONDON EDITIONS OF THAT WORK:

CONTAINING

Correct Solutions

TO

ALL THE QUESTIONS.

THE WHOLE RENDERED AS PLAIN AS THE PRESENT STATE OF THE SCIENCE WILL ADMIT.

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CONTENTS.

A KEY

TO

THE LAST NEW-YORK EDITION

OF

BONNYCASTLE'S ALGEBRA.

Practical Examples for computing the numeral values of various Algebraic Expressions, or combinations of letters.

REQUIRED the numeral values of the following quantities: supposing $a=6$, $b=5$, $c=4$, $d=1$, and $e=0$.

1. $2a^2+3bc-5d=72+60-5=127$.
2. $5a^2b-10ab^2+2e=900-1500+0=-600$.
3. $7a^2+b-c\times d+e=252+5-4+0=253$.
4. $5\sqrt{ab}+b^2-2ab-e^2=5\sqrt{30}+25-60-0=-7.613871$.
5. $\frac{a}{c}\times d-\frac{a-b}{d}+2a^2e=\frac{6}{4}-\frac{1}{1}+0=\frac{2}{4}$, or $\frac{1}{2}$.
6. $2\sqrt{c}+2\sqrt{(2a+b-d)}=6+2\sqrt{16}=14$.
7. $a\sqrt{(a^2+b^2)}+3bc\sqrt{(a^2-b^2)}=6\sqrt{61}+60\sqrt{11}=$ 245.8589862.
8. $3a^2b+\sqrt[3]{(c^2+\sqrt{[2ac+c^2)]}}=540+\sqrt[3]{(16+\sqrt{64})}=$ $540+\sqrt[3]{(16+8)}=540+\sqrt[3]{24}=542.8844991$.
9. $\frac{2b+c}{3a-c}-\frac{\sqrt{5b}+3\sqrt{c}+d}{2a+c}=\frac{10+4}{18-4}-\frac{\sqrt{25}+3\sqrt{4}+1}{16}=$ $\frac{14}{14}-\frac{5+6+1}{16}=1-\frac{12}{16}=1-\frac{3}{4}=\frac{1}{4}$.

ADDITION.

EXAMPLES FOR PRACTICE.

Ex. 1.
$$\begin{array}{l} \frac{1}{2}a+\frac{1}{2}b \\ \frac{1}{2}a-\frac{1}{2}b \\ \hline a \end{array}$$

Ex. 2.
$$\begin{array}{r} 5x-3a+b+7 \\ -3x-4a+2b-9 \\ \hline 2x-7a+3b-2 \end{array}$$

Ex. 3.
$$\begin{array}{l} 2a+3b-4c-9 \\ 5a-3b+2c-10 \\ \hline 7a\quad -2c-19 \end{array}$$

Ex. 4.
$$\begin{array}{r} 3a+2b-5 \\ a+5b-c \\ 6a-2c+3 \\ \hline 10a+7b-3c-2 \end{array}$$

Ex. 5.
$$\begin{array}{l} x^3+ax^2+bx+2 \\ x^3+cx^2+dx-1 \\ \hline 2x^3+(a+c)x^2+(b+d)x+1 \end{array}$$

Ex. 6.
$$\begin{array}{r} 6xy-12x^2 \\ 3xy-4x^2 \\ -2xy+4x^2 \\ -3xy+4x^2 \\ \hline 4xy-8x^2 \end{array}$$

Ex. 7.
$$\begin{array}{r} 4ax+3x^{\frac{1}{2}}-130 \\ 3ax+5x^2+9x^2 \\ 7xy-4x^{\frac{1}{2}}+90 \\ -6x^2+x^{\frac{1}{2}}+40 \\ \hline 7ax+8x^2+7xy \end{array}$$

Ex. 8.
$$\begin{array}{l} 2a^2-3ab+2b^3-3a^2 \\ -2a^2+a^3+3b^3-5c^3 \\ 100+5ab-2b^3+4c^3 \\ 16a^2+20ab-bc-80 \\ \hline 13a^2+22ab+3b^3+a^3-c^3+20-bc \end{array}$$

Ex. 9. $\frac{5a}{b}-\frac{3c^2}{a}+\frac{7\sqrt{bc}}{x}-9\left(\frac{ab+x}{d}\right)$

$\frac{8a}{b}+\frac{7c^2}{a}-\frac{12\sqrt{bc}}{x}+6\left(\frac{ab+x}{d}\right)$

$\frac{13a}{b}+\frac{4c^2}{a}-\frac{5\sqrt{bc}}{x}-3\left(\frac{ab+x}{d}\right)$

Ex. 10.

$$\begin{array}{r} 4bc+3a^2-\ \ e^2+10 \\ 6bc-5a^2+\ 2e^2-15 \\ -9bc-4a^2-10e^2+21 \\ \hline bc-6a^2-\ 9e^2+16 \end{array}$$

SUBTRACTION.

EXAMPLES FOR PRACTICE.

Ex. 1.

$$\begin{array}{r} \frac{1}{2}a+\frac{1}{2}b \\ \frac{1}{2}a-\frac{1}{2}b \\ \hline +b \end{array}$$

Ex. 2.

$$\begin{array}{r} 3x-2a-b+7 \\ 4x+a-3b+8 \\ \hline -x-3a+2b-1 \end{array}$$

Ex. 3.

$$\begin{array}{r} 3a+\ b+\ c-2d \\ -8\ +\ b-8c+2d \\ \hline 8\ +3a+9c-4d \end{array}$$

Ex. 4.

$$\begin{array}{r} 13x^2-2ax+\ 9b^2 \\ 5x^2-7ax-\ \ b^2 \\ \hline 8x^2+5ax+10b^2 \end{array}$$

Ex. 5.

$$\begin{array}{r} 20ax-\ 5\sqrt{x}+3a \\ 4ax+\ 5\sqrt{x}-\ a \\ \hline 16ax-10\sqrt{x}+4a \end{array}$$

Ex. 6.

$$\begin{array}{r} 5ab+2b^2-c+bc-b \\ -2ab+\ b^2\quad\ +bc \\ \hline 7ab+\ b^2-c-b \end{array}$$

Ex. 7.

$$\begin{array}{r} ax^3-bx^2+cx-\ d \\ +bx^2+ex-2d \\ \hline ax^3-2bx^2+(c-e)x+d \end{array}$$

Ex. 8.

$$\begin{array}{r} -6a+13x-4b-12c \\ -9a+\ 4x+4b-\ 5c \\ \hline 3a+\ 9x-8b-\ 7c \\ \hline \end{array}$$

Ex. 9.

$$\begin{array}{r} 6x^2y-3\sqrt{xy}-6ay \\ 3x^2y+3\sqrt{xy}-4ay \\ \hline 3x^2y-6\sqrt{xy}-2ay \\ \hline \end{array}$$

Ex. 10. $4ax-150+\ 4x^{\frac{1}{2}}$ and $2ax-80+7x^2$

$$\begin{array}{r} 4ax-150+\ 4x^{\frac{1}{2}} \\ 3ax+5x^2+10x^{\frac{1}{2}} \\ -2ax+90\ -12x^{\frac{1}{2}} \\ \hline \end{array} \qquad \begin{array}{r} 2ax-80+7x^2 \\ -8ax-70+7x^{\frac{1}{2}} \\ 4a^2x^2+30-4x^{\frac{1}{2}}-2x^2 \\ \hline \end{array}$$

Then, $5ax-60+5x^2+2x^{\frac{1}{2}}$ $-6ax-120+3x^{\frac{1}{2}}+5x^2+4a^2x^2$

$$\begin{array}{r} 5ax-60+5x^2+2x^{\frac{1}{2}} \\ -6ax-120+5x^2+3x^{\frac{1}{2}}+4a^2x^2 \\ \hline \end{array}$$

Ans. $11ax+60-x^{\frac{1}{2}}-4a^2x^2$

MULTIPLICATION.

CASE III.

When both the factors are compound quantities.

EXAMPLES FOR PRACTICE.

Ex. 1.

$$\begin{array}{l} x^2-xy+y^2 \\ x+y \\ \hline x^3-x^2y+xy^2 \\ \quad +x^2y-xy^2+y^3. \\ \hline x^3 \quad * \quad * \quad +y^3 \\ \hline \end{array}$$

Ex. 2.

$$\begin{array}{l} x^3+x^2y+xy^2+y^3 \\ x-y \\ \hline x^4+x^3y+x^2y^2+xy^3 \\ \quad -x^3y-x^2y^2-xy^3-y^4 \\ \hline x^4 \quad * \quad * \quad * \quad -y^4 \\ \hline \end{array}$$

Ex. 3.
$$\begin{array}{l}
x^2+xy+y^2 \\
x^2-xy+y^2 \\
\hline
x^4+x^3y+x^2y^2 \\
\quad -x^3y-x^2y^2-xy^3 \\
\qquad\qquad +x^2y^2+xy^3+y^4 \\
\hline
x^4 \quad * \quad +x^2y^2 \quad * \quad +y^4 \\
\hline
\end{array}$$

Ex. 4.
$$\begin{array}{l}
3x^2-2xy+5 \\
\;x^2+2xy-3 \\
\hline
3x^4-2x^3y+5x^2 \\
\quad +6x^3y-4x^2y^2+10xy \\
\qquad\qquad -9x^2+6xy-15 \\
\hline
3x^4+4x^3y-4x^2y^2+16xy-4x^2-15 \\
\hline
\end{array}$$

Ex. 5.
$$\begin{array}{l}
2a^2-3ax+4x^2 \\
5a^2-6ax-2x^2 \\
\hline
10a^4-15a^3x+20a^2x^2 \\
\quad -12a^3x+18a^2x^2-24ax^3 \\
\qquad\qquad -\;4a^2x^2+6ax^3-8x^4 \\
\hline
10a^4-27a^3x+34a^2x^2-18ax^3-8x^4 \\
\hline
\end{array}$$

Ex. 6.
$$\begin{array}{l}
5x^3+4ax^2+3a^2x+a^3 \\
2x^2-3ax\;+a^2 \\
\hline
10x^5+\;8ax^4+\;6a^2x^3+2a^3x^2 \\
\quad -15ax^4-12a^2x^3-9a^3x^2-3a^4x \\
\qquad\qquad +\;5a^2x^3+4a^3x^2+3a^4x+a^5 \\
\hline
10x^5-\;7ax^4-\;\;a^2x^3-3a^3x^2+\qquad\qquad a^5 \\
\hline
\end{array}$$

Ex. 7. $3x^3+2x^2y^2+3y^3$
$2x^3-3x^2y^2+5y^3$

$$\begin{array}{l} 6x^6+4x^5y^2+6x^3y^3 \\ \quad -9x^5y^2-6x^4y^4-9x^2y^5 \\ \qquad\qquad +15x^3y^3+10x^2y^5+15y^6 \\ \hline 6x^6-5x^5y^2-6x^4y^4+21x^3y^3+x^2y^5+15y^6 \end{array}$$

Ex. 8. x^3-ax^2+bx-c
x^2-dx+e

$$\begin{array}{l} x^5-ax^4+bx^3-cx^2 \\ \quad -dx^4+adx^3-dbx^2+dcx \\ \qquad\quad +ex^3-aex^2+bex-ec \\ \hline x^5-(a+d)x^4+(b+ad+e)x^3-(c+da+ae)x^2+ \\ \quad (db+be)x-ec \end{array}$$

Ex. 9. 1st. Multiply the factors I. and II.

a^2+ab+b^2
$a+b$

$$\begin{array}{l} a^3+a^2b+ab^2 \\ \quad +a^2b+ab^2+b^3 \\ \hline a^3+2a^2b+2ab^2+b^3 \end{array}$$

Next multiply the factors III. and IV.

a^2-ab+b^2
$a-b$

$$\begin{array}{l} a^3-a^2b+ab^2 \\ \quad -a^2b+ab^2-b^3 \\ \hline a^3-2a^2b+2ab^2-b^3 \end{array}$$

It now remains to multiply the first product I. II. by the second product III. IV.

$$
\begin{array}{l}
a^3+2a^2b+2ab^2+b^3 \\
a^3-2a^2b+2ab^2-b^3 \\
\hline
a^6+2a^5b+2a^4b^2+a^3b^3 \\
\quad -2a^5b-4a^4b^2-4a^3b^3-2b^2b^4 \\
\qquad\qquad +2a^4b^2+4a^3b^3+4a^2b^4+2ab^5 \\
\qquad\qquad\qquad -\ a^3b^3-2a^2b^4-2ab^5-b^6 \\
\hline
a^6 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -b^6
\end{array}
$$

2d. Change the order of the question, that is, multiply the factors I. and III., then II. and IV. together.

$$
\begin{array}{l}
a+b \\
a-b \\
\hline
a^2+ab \\
\quad -ab-b^2 \\
\hline
a^2 \quad - \quad b^2
\end{array}
$$

Then,

$$
\begin{array}{l}
a^2+ab+b^2 \\
a^2-ab+b^2 \\
\hline
a^4+a^3b+a^2b^2 \\
\quad -a^3b-a^2b^2-ab^3 \\
\qquad\quad +a^2b^2+ab^3+b^4 \\
\hline
a^4 \qquad +a^2b^2 \qquad +b^4
\end{array}
$$

Then multiply the products I. III. and II. IV.

$$
\begin{array}{l}
a^4+a^2b^2+b^4 \\
a^2-b^2 \\
\hline
a^6+a^4b^2+a^2b^4 \\
\quad -a^4b^2-a^2b^4-b^6 \\
\hline
a^6 \qquad\qquad\qquad -b^6
\end{array}
$$

which is the product.

3d. Again, multiply the I. factor by the IV., and next, the II. by the III.

$$
\begin{array}{l}
a^2-ab+b^2 \\
a+b \\
\hline
a^3-a^2b+ab^2 \\
\quad +a^2b-ab^2+b^3 \\
\hline
a^3 \qquad\qquad\quad +b^3
\end{array}
$$

next,

$$
\begin{array}{l}
a^2+ab+b^2 \\
a-b \\
\hline
a^3+a^2b+ab^2 \\
\quad -a^2b-ab^2-b^3 \\
\hline
a^3 \qquad\qquad\quad -b^3
\end{array}
$$

It remains to multiply the product I. IV. and II. III.

$$\begin{array}{l}
a^3+b^3 \\
a^3-b^3 \\
\hline
a^6+a^3b^3 \\
\quad -a^3b^3-b^6 \\
\hline
a^6 \quad * \quad -b^6
\end{array}$$

as in the foregoing cases.

It will now be proper to illustrate this example by a numerical application.

Suppose $a=3$, and $b=2$. We shall have $a+b=5$, and $a-b=1$: further, $a^2=9$, $ab=6$, and $b^2=4$. Therefore, $a^2+ab+b^2=19$, and $a^2-ab+b^2=7$. So that the product required is that of $5\times19\times1\times7=665$.

Now, $a^6=729$, and $b^6=64$; consequently the product is $a^6-b^6=665$, as we have already seen.

Ex. 10.
$$\begin{array}{lllllll}
a^3 & +3a^2x & +3ax^2 & +x^3 & & & \\
a^3 & -3a^2x & +3ax^2 & \quad x^3 & & & \\
\hline
a^6 & +3a^5x & +3a^4x^2 & +a^3x^3 & & & \\
 & -3a^5x & -9a^4x^2 & -9a^3x^3 & -3a^2x^4 & & \\
 & & +3a^4x^2 & +9a^3x^3 & +9a^2x^4 & +3ax^5 & \\
 & & & -\ a^3x^3 & -3a^2x^4 & -3ax^5 & -x^6 \\
\hline
a^6 & * & -3a^4x^2 & * & +3a^2x^4 & * & -x^6 \\
\hline
\end{array}$$

Ex. 11.
$$\begin{array}{llll}
a^4 & +a^2c^2 & +c^4 & \\
a^2 & -c^2 & & \\
\hline
a^6 & +a^4c^2 & +a^2c^4 & \\
 & -a^4c^2 & -a^2c^4 & -c^6 \\
\hline
a^6 & & & -c^6 \\
\hline
\end{array}$$

Ex. 12.
$$\begin{array}{l}
a^2+b^2+c^2-ab-ac-bc \\
a+b+c \\
\hline
a^3+ab^2+ac^2-a^2b-a^2c-abc \\
\quad +a^2b+b^3+bc^2-ab^2-abc-b^2c \\
\qquad\qquad a^2c+b^2c+c^3-abc-ac^2-bc^2 \\
\hline
a^3 \quad * \quad +b^3 \quad * \quad -3abc+c^3 \\
\hline
\end{array}$$

DIVISION.

CASE I.

When the divisor and dividend are both simple quantities.

Ex. 1. $16x^2 \div 8x$, or $\frac{16x^2}{8x}=2x$; and $\frac{12a^2x^2}{-8a^2x}=-\frac{3x}{2}$.

Ex. 2. $-15ay^2 \div 3ay$, or $\frac{-15ay^2}{3ay}=-5y$; and $\frac{-18ax^2y}{-8ax}$ $=2\frac{1}{4}xy$.

Ex. 3. $-\frac{2}{3}a^{\frac{1}{2}} \div \frac{1}{5}a^{\frac{1}{2}}=\frac{-2}{3}\times\frac{5}{1}=-3\frac{1}{3}$; and $ax^{\frac{1}{3}} \div -\frac{3}{5}$ $a^{\frac{1}{2}}x^{\frac{1}{4}}=-\frac{5}{3}a^{\frac{1}{2}}x^{\frac{1}{12}}$.

Ex. 4. $12a^2b^2 \div -3a^2b$, or $-\frac{12a^2b^2}{3a^2b}=-4b$; and $-$ $15ay^{\frac{2}{3}} \div -3ay^{\frac{1}{2}}$, or $+\frac{15ay^{\frac{2}{3}}}{3ay^{\frac{1}{2}}}=5y^{\frac{2}{3}-\frac{1}{2}}=5y^{\frac{1}{6}}$.

Ex. 5. $-15a^2x^2 \div 5ax^2$, or $-\frac{15a^2x^2}{5ax^2}=-3a$; and $21a^2c^2x^{\frac{1}{2}} \div -7ac^2x^{\frac{1}{4}}$, or $-\frac{21a^2c^2x^{\frac{1}{2}}}{7ac^2x^{\frac{1}{4}}}=-3ax^{\frac{1}{4}}$.

Ex. 6. $-17x^{\frac{1}{2}}a^3c \div -5x^{\frac{1}{3}}a^2c^{\frac{1}{2}}$, or $\frac{17x^{\frac{1}{2}}a^3c}{5x^{\frac{1}{3}}a^2c^{\frac{1}{2}}} = \ldots\ldots$

$\frac{17x^{\frac{1}{2}-\frac{1}{3}}ac^{1-\frac{1}{2}}}{5} = \frac{17x^{\frac{1}{6}}ac^{\frac{1}{2}}}{5}$; and $24\sqrt{xy} \div 8\sqrt{xy}$, or $\frac{24xy}{8\sqrt{xy}} \ldots$ $= 3\sqrt{xy}$.

CASE II.

When the divisor is a simple quantity, and the dividend a compound one.

Ex. 1. Here $\frac{3x^3+6x^2+3ax-15x}{3x} = x^2+2x+a-5$.

Ex. 2. Here $\frac{3abc+12abx-9a^2b}{3ab} = c+4x-3a$.

Ex. 3. Here $\frac{40a^3b^3+60a^2b^2-17ab}{-ab} = -40a^2b^2-60ab+17$.

Ex. 4. Here $\frac{15a^2bc-12acx^2+5ad^2}{5ac} = 3ab-2\frac{2}{5}x^2+\frac{d^2}{c}$.

Ex. 5. Here $\frac{20ax^3+15ax^2+10ax+5a}{5a} = 4x^3+3x^2+2x+1$.

Ex. 6. Here $\frac{6bcdz+4bzd^2-2b^2z^2}{2bz} = 3cd+2d^2-bz$.

Ex. 7. Here $\frac{14a^2-7ab+21ax-28a}{7a} = 2a-b+3x-4$.

Ex. 8. Here $\frac{-20ab+60ab^3-12a^2b^2}{-4ab} = 5-15b^2+3ab$.

CASE III.

When the divisor and dividend are both compound quantities.

$$
\begin{array}{l}
a-x)a \;.\;.\;.\;.\; (1+\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}+\frac{x^4}{a^4}+, \&c. \\
\quad\;\; \underline{a-x} \\
\quad\;\; x \\
\quad\;\; \underline{x-\frac{x^2}{a}} \\
\qquad\quad \frac{x^2}{a} \\
\qquad\quad \underline{\frac{x^2}{a}-\frac{x^3}{a^2}} \\
\qquad\qquad\quad \frac{x^3}{a^2} \\
\qquad\qquad\quad \underline{\frac{x^3}{a^2}-\frac{x^4}{a^3}} \\
\qquad\qquad\qquad\quad \frac{x^4}{a^3} \\
\qquad\qquad\qquad\quad \underline{\frac{x^4}{a^3}-\frac{x^5}{a^4}}
\end{array}
$$

EXAMPLES FOR PRACTICE.

Ex. 1.
$$
\begin{array}{l}
a-x)a^2-2ax+x^2(a-x \\
\qquad\;\; \underline{a^2-\;ax} \\
\qquad\qquad -\;ax+x^2 \\
\qquad\qquad \underline{-\;ax+x^2}
\end{array}
$$

Ex. 2.

$$
\begin{array}{l}
x-a)\,x^3-3ax^2+3a^2x-a^3\,(x^2-2ax+a^2 \\
\qquad x^3-\ ax^2 \\
\hline
\qquad\quad -2ax^2+3a^2x \\
\qquad\quad -2ax^2+2a^2x \\
\hline
\qquad\qquad\qquad a^2x-a^3 \\
\qquad\qquad\qquad a^2x-a^3 \\
\hline
\end{array}
$$

Ex. 3.

$$
\begin{array}{l}
a+x)\,a^3+5a^2x+5ax^2+x^3\,(a^2+4ax+x^2 \\
\qquad a^3+\ a^2x \\
\hline
\qquad\quad 4a^2x+5ax^2 \\
\qquad\quad 4a^2x+5ax^2 \\
\hline
\qquad\qquad\qquad ax^2+x^3 \\
\qquad\qquad\qquad ax^2+x^3 \\
\hline
\end{array}
$$

Ex. 4.

$$
\begin{array}{l}
y-8)\,2y^3-19y^2+26y-16\,(2y^2-3y+2 \\
\qquad 2y^3-16y^2 \\
\hline
\qquad\quad -\ 3y^2+26y \\
\qquad\quad -\ 3y^2+24y \\
\hline
\qquad\qquad\qquad 2y-16 \\
\qquad\qquad\qquad 2y-16 \\
\hline
\end{array}
$$

Ex. 5.

$$
\begin{array}{l}
x+1)\,x^5+1\,(x^4-x^3+x^2-x+1 \\
\qquad x^5+x^4 \\
\hline
\qquad\quad -x^4+1 \\
\qquad\quad -x^4-x^3 \\
\hline
\qquad\qquad\quad x^3+1 \\
\qquad\qquad\quad x^3+x^2 \\
\hline
\qquad\qquad\qquad -x^2+1 \\
\qquad\qquad\qquad -x^2-x \\
\hline
\qquad\qquad\qquad\qquad x+1 \\
\qquad\qquad\qquad\qquad x+1 \\
\end{array}
$$

Again. $x-1)x^6-1(x^5+x^4+x^3+x^2+x+1$

$$\begin{array}{l}
x^6-x^5 \\
\hline
x^5-1 \\
x^5-x^4 \\
\hline
x^4-1 \\
x^4-x^3 \\
\hline
x^3-1 \\
x^3-x^2 \\
\hline
x^2-1 \\
x^2-x \\
\hline
x-1 \\
x-1 \\
\hline
\end{array}$$

Ex. 6.

$2x-3a)48x^3-76ax^2-64a^2x+105a^3(24x^2-2ax-35a^2$

$$\begin{array}{l}
48x^3-72ax^2 \\
\hline
-4ax^2-64a^2x \\
-4ax^2+6a^2x \\
\hline
-70a^2x+105a^3 \\
-70a^2x+105a^3 \\
\hline
\end{array}$$

Ex. 7. $2x^2+3x-1)4x^4-9x^2+6x-1(2x^2-3x+1$

$$\begin{array}{l}
4x^4+6x^3-2x^2 \\
\hline
-6x^3-7x^2+6x \\
-6x^3-9x^2+3x \\
\hline
2x^2+3x-1 \\
2x^2+3x-1 \\
\hline
\end{array}$$

Ex. 8. $x^2-ax+a^2)x^4-a^2x^2+2a^3x-a^4(a^2+ax-a^2$

$$x^4-a\,x^3+a^2x^2$$

$$ax^3-2a^2x^2+2a^3x$$
$$ax^3-\ a^2x^2+\ a^3x$$

$$-a^2x^2+a^3x-a^4$$
$$-a^2x^2+a^3x-x^4$$

Ex. 9. $3x-6)6x^4-96\ (2x^3+4x^2+8x+16$

$$6x^4-12x^3$$

$$12x^3-96$$
$$12x^3-24x^2$$

$$24x^2-96$$
$$24x^2-48x$$

$$48x-96$$
$$48x-96$$

Again. $a+x)a^5+x^5\ (a^4-a^3x+a^2x^2-ax^3+x^4$

$$a^5+a^4x$$

$$-a^4x+x^5$$
$$-a^4x-a^3x^2$$

$$a^3x^2+x^5$$
$$a^3x^2+a^2x^3$$

$$-a^2x^3+x^5$$
$$-a^2x^3-ax^4$$

$$ax^4+x^5$$
$$ax^4+x^5$$

Ex. 10. $2x+3)32x^5+243(16x^4-24x^3+36x^2-54x+81$

$$32x^5+48x^4$$

$$-48x^4+243$$

$$-48x^4-72x^3$$

$$72x^3+243$$

$$72x^3+108x^2$$

$$-108x^2+243$$

$$-108x^2-162x$$

$$162x+243$$

$$162x+243$$

Again. $x-a)x^6-a^6\ (x^5+ax^4+a^2x^3+a^3x^2+a^4x+a^5$

$$x^6-ax^5$$

$$ax^5-a^6$$

$$ax^5-a^2x^4$$

$$a^2x^4-a^6$$

$$a^2x^4-a^3x^3$$

$$a^3x^3-a^6$$

$$a^3x^3-a^4x^2$$

$$a^4x^2-a^6$$

$$a^4x^2-a^5x$$

$$a^5x-a^6$$

$$a^5x-a^6$$

Ex. 11. $b-y)b^4-3y^4\left(b^3+b^2y+by^2+y^3-\frac{2y^4}{b-y}\right.$

$$\begin{array}{l}
b^4-b^3y \\
\hline
\quad b^3y-3y^4 \\
\quad b^3y-b^2y^2 \\
\hline
\qquad b^2y^2-3y^4 \\
\qquad b^2y^2-by^3 \\
\hline
\qquad\quad by^3-3y^4 \\
\qquad\quad by^3-\ y^4 \\
\hline
\qquad\qquad -2y^4 \quad \text{remainder.}
\end{array}$$

Again.

$a+2b)a^4+4a^2b+8b^4\left(a^3-2a^2b+4ab+4ab^2-(8b^2+8b^3)\right)$

$$\begin{array}{l}
a^4+2a^3b \\
\hline
\quad -2a^3b+4a^2b \\
\quad -2a^3b-4a^2b^2 \\
\hline
\qquad +4a^2b+4a^2b^2 \\
\qquad +4a^2b+8a\ b^2 \\
\hline
\qquad\quad +4a^2b^2-\ 8ab^2 \\
\qquad\quad +4a^2b^2+\ 8ab^3 \\
\hline
\qquad\quad\ -8ab^3-\ 8ab^3 \\
\qquad\quad\ -8ab^3-16\ b^3 \\
\hline
\qquad\qquad -8ab^3+16b^3+8b^4 \\
\qquad\qquad -8ab^2-16b^4 \\
\hline
\qquad\qquad\quad +16b^3+24b^4 \\
\hline
\end{array}$$

Ex. 12.

$$
\begin{array}{l}
x+a)\,x^2+px+q\,\left(x+(p-a)-\dfrac{pa}{x}+\dfrac{pa^2}{x^2}+\&c.\right. \\
\qquad\qquad x^2+ax \\
\qquad\qquad \overline{(p-a)x+q} \\
\qquad\qquad (p-a)x+pa-a^2 \\
\qquad\qquad\qquad \overline{-pa+a^2+q} \\
\qquad\qquad\qquad -pa-\dfrac{pa^2}{x} \\
\qquad\qquad\qquad\qquad \overline{+\dfrac{pa^2}{x}+a^2+q} \\
\qquad\qquad\qquad\qquad +\dfrac{pa^2}{x}+\dfrac{pa^2}{x^2} \\
\qquad\qquad\qquad\qquad\qquad \overline{\&c.\ \&c.}
\end{array}
$$

Again,

$$
\begin{array}{l}
x-a)\,x^3-px^2+qx-r\,(x^2+(a-p)x+(a^2-ap+q)+\&c. \\
\qquad x^3-ax^2 \\
\qquad\qquad \overline{(a-p)x^2+qx} \\
\qquad\qquad (a-p)x^2-(a^2-ap)x \\
\qquad\qquad\qquad \overline{(a^2-ap+q)x} \\
\qquad\qquad\qquad (a^2-ap+q)x-a(a^2-ap+q) \\
\qquad\qquad\qquad\qquad \overline{+a^3-a^2p+aq\ \&c.\ \&c.}
\end{array}
$$

Ex. 13.

$$
\begin{array}{l}
1-2x+x^2)\,1-5x+10x^2-10x^3+5x^4-x^5\,(1-3x+3x^2-x^3 \\
\qquad\qquad 1-2x+x^2 \\
\qquad\qquad \overline{-3x+9x^2-10x^3} \\
\qquad\qquad -3x+6x^2-3x^3 \\
\qquad\qquad\qquad \overline{3x^2-7x^3+5x^4} \\
\qquad\qquad\qquad 3x^2-6x^3+3x^4 \\
\qquad\qquad\qquad\qquad \overline{-x^3+2x^4-x^5} \\
\qquad\qquad\qquad\qquad -x^3+2x^4-x^5
\end{array}
$$

Ex. 14. $a^2-2ab+2b^2)a^4+4b^4(a^2+2ab+2b^2$

$$\begin{array}{r}
a^4-2a^3b+2a^2b^2 \\
\hline
2a^3b-2a^2b^2+4b^4 \\
2a^3b-4a^2b^2+4ab^3 \\
\hline
2a^2b^2-4ab^3+4b^4 \\
2a^2b^2-4ab^3+4b^4 \\
\hline
\end{array}$$

Ex. 15.

$a^2-2ax+x^2)a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5(a^3$

$$\begin{array}{r}
a^5-2a^4x+\ \ a^3x^2 \qquad [-3a^2x+3ax^2-x^3 \\
\hline
-3a^4x+9a^3x^2-10a^2x^3 \\
-3a^4x+6a^3x^2-\ \ 3a^2x^3 \\
\hline
3a^3x^2-7a^2x^3+5ax^4 \\
3a^3x^2-6a^2x^3+3ax^4 \\
\hline
-a^2x^3+2ax^4-x^5 \\
-a^2x^3+2ax^4-x^5 \\
\hline
\end{array}$$

Ex. 16. $a^2+ab\sqrt{2}+b^2)a^4+b^4\quad(a^2-ab\sqrt{2}+b^2$

$$\begin{array}{r}
a^4+a^3b\sqrt{2}+a^2b^2 \\
\hline
-a^3b\sqrt{2}-\ \ a^2b^2 \\
-a^3b\sqrt{2}-2a^2b^2-ab^3\sqrt{2} \\
\hline
a^2b^2+ab^3\sqrt{2}+b^4 \\
a^2b^2+ab^3\sqrt{2}+b^4 \\
\hline
\end{array}$$

ALGEBRAIC FRACTIONS.

CASE I.

To find the greatest common measure of the terms of a fraction.

Ex. 4. Here $x^3-a^3)x^4-a^4(x$

$$
\begin{array}{l}
x^4-a^3x \\
\hline
a^3x-a^4
\end{array}
$$

Divide by a^3, then $x-a)x^3-a^3(x^2+ax+a^2$

$$
\begin{array}{l}
x^3-ax^2 \\
\hline
ax^2-a^3 \\
ax^2-a^2x \\
\hline
a^2x-a^3 \\
a^2x-a^3 \\
\hline
\end{array}
$$

Therefore $x-a$ is the greatest common measure sought.

Ex. 5. Here $a^3-a^2x-ax^2+x^3)a^4-x^4 \quad . \quad . \quad . \quad (a$

$$
\begin{array}{l}
a^4-a^3x-a^2x^2+ax^3 \\
a^3x+a^2x^2-ax^3-x^4
\end{array}
$$

Dividing the remainder by x

$a^3+a^2x-ax^2-x^3)a^3-a^2x-ax^2+x^3(1$

$$
\begin{array}{l}
a^3+a^2x-ax^2-x^3 \\
\hline
-2a^2x+2x^3
\end{array}
$$

Divide by $2x$; $-a^2+x^2)a^3+a^2x-ax^2-x^3(-a$

$$
\begin{array}{l}
a^3-ax^2 \\
\hline
a^2x-x^3
\end{array}
$$

Divide by x; $a^2-x^2)-a^2+x^2(-1$

$$
\begin{array}{l}
-a^2+x^2 \\
\hline
\end{array}
$$

Therefore a^2-x^2 is the greatest common measure sought.

It frequently happens that the common measure of quantities of this kind is better found by resolving both

numerator and denominator into their component factors, in which it will be useful to remember that the difference of any two even powers is divisible both by the difference and sum of their roots; and that the difference of two odd powers is divisible by the difference of their roots, and their sum by the sum of their roots: thus, in the last example,

$$\frac{a^4-x^4}{a^3-a^2x-ax^2+x^3}=\frac{(a^2+x^2)(a^2-x^2)}{(a^2-x^2)a-(a^2-x^2)x}$$

where it is obvious that a^2-x^2 is a common measure of both terms; and dividing by it, the fraction reduces to $\frac{a^2+x^2}{a-x}$, which is in its lowest terms; and consequently a^2-x^2 is the greatest common measure, as was found by the former rule.

And the same method may be advantageously used in other examples of this kind.

Ex. 6.

$$\begin{array}{l} \text{Here } x^4+a^2x^2+a^4)\,x^4+ax^3-a^3x-a^4\,(1 \\ \qquad\qquad\qquad\qquad\quad x^4+a^2x^2+a^4 \\ \hline \end{array}$$

$$\begin{array}{l} \text{Divide} \qquad\qquad\qquad\qquad ax^3-a^2x^2-a^3x-2a^4 \\ \text{by } a:\ x^3-ax^2-a^2x-2a^3)\,x^4+a^2x^2+a^4\,(x+a \\ \qquad\qquad\qquad\qquad\qquad\qquad x^4-ax^3-a^2x^2-2a^3x \\ \hline \qquad\qquad\qquad\qquad\qquad\qquad\quad ax^3+2a^2x^2+2a^3x+a^4 \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad ax^3-\ a^2x^2-\ a^3x-2a^4 \\ \hline \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 3a^2x^2+3a^3x+3a^4 \end{array}$$

$$\begin{array}{l} \text{Divide by } 3a^2\text{, and we have} \\ \quad x^2+ax+a^2)\,x^3-ax^2-a^2x-2a^3\,(x-2a \\ \qquad\qquad\qquad\quad x^3+ax^2+a^2x \\ \hline \qquad\qquad\qquad\quad -2ax^2-2a^2x-2a^3 \\ \qquad\qquad\qquad\quad -2ax^2-2a^2x-2a^3 \\ \hline \end{array}$$

Therefore x^2+ax+a^2 is the greatest common measure sought.

Ex. 7. Multiplying the denominator by 7, we have

$$7a^2-23ab+6b^2)35a^3-126a^2b+77ab^2-42b^3(5a$$
$$35a^3-115a^2b+30ab^2$$
$$-11a^2b+47ab^2-42b^3$$

Multiply by 7

$$-77a^2b+329ab^2-294b^3(-11b$$
$$-77a^2b+253ab^2-66b^3$$
$$76ab^2-228b^3$$

Dividing by $76b^2$, and we have

$$a-3b)7a^2-23ab+6b^2(7a-2b$$
$$7a^2-21ab$$
$$-2ab+6b^2$$
$$-2ab+6b^2$$

Therefore $a-3b$ is the greatest common measure sought.

Ex. 8. $\dfrac{x^3+ax^2+bx^2-2a^2x+bax-2ba^2}{x^2-bx+2ax-2ab}=$ . . . .

$\dfrac{x^3+ax^2+bx^2-2a^2x+bax-2ba^2}{(x+2a)\times(x-b)}$. Here I divide the numerator by $x+2a$, and the quotient comes out exactly; therefore $x+2a$ is the greatest common measure sought.

Ex. 9.

$$x^3-ax^2-8a^2x+6a^3)x^4-3ax^3-8a^2x^2+18a^3x-8a^4(x-2a$$
$$x^4-ax^3-8a^2x^2+6a^3x$$
$$-2ax^3+12a^3x-8a^4$$
$$-2ax^3+2a^2x^2+16a^3x-12a^4$$
$$-2a^2x^2-4a^3x+4a^4$$

Dividing by $-2a^2$, and we have

$$x^2+2ax-2a^2)x^3-ax^2-8a^2x+6a^3(x-3a$$
$$x^3+2ax^2-2a^2x$$
$$-3ax^2-6a^2x+6a^3$$
$$-3ax^2-6a^2x+6a^3$$

Therefore $x^2+2ax-2a^2$ is the greatest common measure sought.

Ex. 10. Here, dividing the numerator by $5a^3$, and the denominator by b, we have

$$\begin{array}{l} a^2+2ab+b^2)a^3+2a^2b+2ab^2+b^3(a \\ \qquad\qquad\quad a^3+2a^2b+ab^2 \\ \hline \qquad\qquad\qquad\qquad ab^2+b^3 \end{array}$$

Dividing by b^2, and we have

$$\begin{array}{l} a+b)a^2+2ab+b^2(a+b \\ \quad\quad a^2+ab \\ \hline \qquad\quad ab+b^2 \\ \qquad\quad ab+b^2 \\ \hline \end{array}$$

Therefore $a+b$ is the greatest common measure sought.

Ex. 12. Here the numerator being the least compounded, and b rising therein to a single dimension only, I divide the same into the parts $6a^5-4a^3c^2$, and $15a^4b-10a^2bc^2$, which, by inspection, appear to be equal to $2a^3(3a^2-2c^2,)$ and $5a^2b(3a^2-3c^2.)$ Therefore, $3a^2-2c^2$ is a divisor to both the parts, and likewise to the whole, expressed by $(3a^2-2c^2)\times(2a^3+5a^2b)$; so that one of these two factors, if the fraction given can be reduced to lower terms, must also measure the denominator; but the former will be found to succeed: thus,

$$\begin{array}{l} 3a^2-2c^2)9a^3b-27a^2bc-6abc^2+18bc^3(3ab-9bc \\ \qquad\qquad 9a^3b-\ 6abc^2 \\ \hline \qquad\qquad\qquad -27a^2bc+18bc^3 \\ \qquad\qquad\qquad -27a^2bc+18bc^3 \\ \hline \end{array}$$

Therefore $3a^2-2c^2$ is the greatest common measure sought.

CASE II.

To reduce fractions to their lowest terms.

Ex. 4. Here $\dfrac{x^4-a^4}{x^5-a^2x^3}=\dfrac{(x^2+a^2)(x^2-a^2)}{x^3(x^2-a^2)}=\dfrac{x^2+a^2}{x^3}$; by dividing both terms by x^2-a^2.

Ex. 5. $6a^2+7ax-3x^2)6a^2+11ax+3x^2(1$
$$6a^2+7ax-3x^2$$
$$4ax+6x^2$$

Divide by $2x$; $2a+3x)6a^2+7ax-3x^2(3a-x$
$$6a^2+9ax$$
$$-2ax-3x^2$$
$$-2ax-3x^2$$

Therefore $2a+3x)\frac{6a^2+7ax-3x^2}{6a^2+11ax+3x^2}=\frac{3a-x}{3a+x}$, the fraction sought.

Ex. 6. Here $\frac{2x^3-16x-6}{3x^3-24x-9}=\frac{2(x^3-8x-3)}{3(x^3-8x-3)}=\frac{2}{3}$.

Ex. 7. Here, multiplying the numerator by 5, and we have

$15x^4-2x^3+10x^2-x+2)45x^5+10x^3+20x^2-5x+4(3x$
$$45x^5-6x^4+30x^3-3x^2+6x$$
$$6x^4-20x^3+23x^2-11x+5$$

Mult. by 5

$15x^4-2x^3+10x^2-x+2)30x^4-100x^3+115x^2-55x+25(+2$
$$30x^4-4x^3+20x^2-2x+4$$
$$-96x^3+95x^2-53x+21$$

Multiply the last divisor by 32, and we shall have

$-96x^3+95x^2-53x+21)480x^4-64x^3+320x^2-32x+64(-5x$
$$480x^4-475x^3+265x^2-105x$$
$$411x^3+55x^2+73x+64$$

Mult. by 32

$-96x^3+95x^2-53x+21)13152x^3+1760x^2+2336x+2048(-137$
$$13152x^3-13015x^2+7261x-2877$$
$$14775x^2-4925x+4925$$

Dividing the latter by 4925, it becomes $3x-x+1$; which by another operation, exactly divides $-96x^3+95x^2-53x+21$; and therefore is the common measure; and the reduced fraction is $\frac{3x^3+x^2+1}{5x^2+x+2}$.

Ex. 8. Here, the denominator being the least compounded, and d rising therein to a single dimension only, I divide the same into the parts $4a^2d-4acd$, and $-2ac^2+2c^3$; which, by inspection, appear to be equal to $4ad(a-c)$, and $-2c^2(a-c)$. Therefore $a-c$ is a divisor to both the parts, and likewise to the whole, expressed by $(4ad-2c^2)\times(a-c)$; so that one of these two factors, if the fraction given can be reduced to lower terms, must also measure the numerator; but the latter will be found to succeed: thus,

$$a-c)a^2d^2-c^2d^2-a^2c^2+c^4(ad^2+cd^2-ac^2-c^3$$
$$a^2d^2-acd^2$$
$$\overline{acd^2-c^2d^2}$$
$$acd^2-c^2d^2$$
$$\overline{-a^2c^2+c^4}$$
$$-a^2c^2+ac^3$$
$$\overline{-ac^3+c^4}$$
$$-ac^3+c^4$$

Therefore, $a-c$ is the greatest common measure;

$$a-c)\frac{a^2d^2-c^2d^2-a^2c^2+c^4}{4a^2d-4acd-2ac^2+2c^3}(=\frac{ad^2+cd^2-ac^2-c^3}{4ad-2c^2}$$

the Ans.

CASE III.

To reduce a mixed quantity to an improper fraction.

Ex. 3. Here $1-\frac{2x}{a}=\frac{1\times a-2x}{a}=\frac{a-2x}{a}$ Ans.

Ex. 4. Here $5a-\frac{3x-b}{a}=\frac{5a^2-(3x-b)}{a}=\frac{5a^2-3x+b}{a}$ the fraction required.

Ex. 5. Here $x-\frac{a+x^2}{2a}=\frac{2ax-(a+x^2)}{2a}=\frac{2ax-a-x^2}{2a}$.

Ex. 6. Here $5+\frac{2x-7}{3x}=\frac{15x+2x-7}{3x}=\frac{17x-7}{3x}$.

Ex. 7. Here $1-\frac{x-a-1}{a}=\frac{a-(x-a-1)}{a}=\frac{a-x+a+1}{a}$
$=\frac{2a-x+1}{a}$ Ans.

Ex. 8. Here $1+2x-\frac{x-3}{5x}=\frac{5x(1+2x)-(x-3)}{5x}=$ . .
$\frac{5x+10x^2-x+3}{5x}=\frac{10x^2+4x+3}{5x}$ Ans.

CASE IV.

To reduce an improper fraction to a whole or mixed quantity.

Ex. 2. Here, $(ax-x^3)\div x=a-x^2$. Ans.

Ex. 3. Here $(ab-2a^2)\div ab=(b-2a)\div b=1-\frac{2a}{b}$.

Ex. 4.
$$\begin{array}{l} a-x)a^2+x^2(a+x \\ \quad\;\; \underline{a^2-ax} \\ \qquad\quad ax+x^2 \\ \qquad\quad \underline{ax-x^2} \\ \qquad\qquad 2x^2 \end{array}$$

Therefore $a+x+\frac{2x^2}{a-x}$ is the mixed number sought.

Ex. 5. Here $\frac{x^3-y^3}{x-y}=x^2+xy+y^2$, as is readily found by division.

Ex. 6. Here $(10x^2-5x+3)\div 5x=2x-1+\frac{3}{5x}$ Ans.

CASE V.

To reduce fractions to other equivalent ones having a common denominator.

Ex. 2. Here by the rule,

$$\left.\begin{aligned} 2x\times c &= 2cx \\ b\times a &= ab \end{aligned}\right\} \text{new numerators.}$$

$$a\times c = ac \quad \textit{common denominator.}$$

Hence $\frac{2xc}{bc}$ and $\frac{ab}{ac}$ are the fractions sought.

Ex. 3. Here

$$\left.\begin{aligned} a\times c &= ac \\ (a+b)b &= ab+b^2 \end{aligned}\right\} \text{new numerators.}$$

$$b\times c = bc \quad \text{common denominator.}$$

Hence $\frac{ac}{bc}$ and $\frac{ab+b^2}{bc}$ are the fractions sought.

Ex. 4. Here

$$\left.\begin{aligned} 3x\times 3c &= 9cx \\ 2b\times 2a &= 4ab \\ d\times 2a\times 3c &= 6acd \end{aligned}\right\} \text{new numerators.}$$

$$1\times 3c\times 2a = 6ac \quad \text{common denominator.}$$

Hence $\frac{9cx}{6ac}$, $\frac{4ab}{6ac}$, and $\frac{6acd}{6ac}$ are the fractions required.

Ex. 5. Here the three fractions, when reduced, are

$$\frac{3}{4}, \frac{2x}{3}, \text{ and } \frac{5a+4x}{5},$$

Therefore,

$$\left.\begin{aligned} 3\times 3\times 5 &= 45 \\ 2x\times 4\times 5 &= 40x \\ (5a+4x)\times 4\times 3 &= 60a+48x \end{aligned}\right\} \text{new numerators.}$$

$$4\times 3\times 5 = 60 \quad \text{common denominator.}$$

The fractions therefore are $\frac{45}{60}$, $\frac{40x}{60}$, and $\frac{60a+48x}{60}$.

Ex. 6. Here

$$\left.\begin{aligned} a\times 7\times (a-x) &= 7a^2-7xa \\ 3x\times 2\times (a-x) &= 6xa-6x^2 \\ 2\times 7\times (a+x) &= 14a+14x \end{aligned}\right\} \text{numerators.}$$

$$2\times 7\times (a-x) = 14a-14x \quad \text{com. denom.}$$

Hence $\frac{7a^2-7xa}{14a-14x}$, $\frac{6xa-6x^2}{14a-14x}$ and $\frac{14a+14x}{14a-14x}$ are the fractions required.

CASE VI.

To add fractional quantities together.

Ex. 4. Here $\frac{2x}{5}+\frac{5x}{7}=\frac{14x}{35}+\frac{25x}{35}=\frac{39x}{35}$ Ans.

Ex. 5. Here $\frac{3x}{2a}+\frac{x}{5}=\frac{15x}{10a}+\frac{2ax}{10a}=\frac{15x+2ax}{10a}$ Ans.

Ex. 6. Here $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=\frac{12x}{24}+\frac{8x}{24}+\frac{6x}{24}=\frac{13x}{12}$

Ex. 7. Here $\frac{4x}{7}+\frac{x-2}{5}=\frac{20x}{35}+\frac{7(x-2)}{35}=\frac{20x+7x-14}{12}$
$\frac{27x-14}{35}$ Ans.

Ex. 8. This may be written $2a+3a+a+\frac{2x}{5}-\frac{8x}{9}=6a$
$+\frac{2x}{5}-\frac{8x}{9}=6a+\frac{18x}{45}-\frac{40x}{45}=6a-\frac{22x}{45}$ Ans.

Ex. 9. Here the fractions are $\frac{3x}{5}, \frac{a}{a-x}$, and $\frac{a-x}{a}$.

By the rule, $\left.\begin{array}{ll} 3x\times(a-x)\times a & =3a^2x-3ax^2 \\ a\times5\times a & =5a^2 \\ (a-x)\times(a-x)5 & =5a^2-10ax+5x^2 \end{array}\right\}$ numerators.

Their sum $=10a^2-10ax+3a^2x-3ax^2+5x^2$, and $5\times(a-x)\times a=5a^2-5ax$ common denominator.

Hence the sum is $2a+\frac{10a^2-10ax+3a^2x-3ax^2+5x^2}{5a^2-5ax}$
$=2a+2+\frac{3a^2x-3ax^2+5x^2}{5a^2-5ax}$ Ans.

Ex. 10. This is the same as $9x+\frac{x-2}{3}-\frac{2x-3}{5x}=9x+$
$\frac{5x^2-10x}{15x}-\frac{6x-9}{15x}=9x+\frac{5x^2-10x-6x+9}{15x}=9x+ \ldots$
$\frac{5x^2-16x+9}{15x}$ Ans.

Ex. 11. Here $5x+\frac{2a}{3x^2}+\frac{a+2x}{4x}=5x+\frac{8ax}{12x^3}+$ ....
$\frac{3ax^2+6x^3}{12x^3}=5x+\frac{8ax+3ax^2+6x^3}{12x^3}=5x+\frac{8a+3ax+6x^2}{12x^2}$.

We have not in all these examples followed exactly the process described in the rule, at least not so as to exhibit the operation, both in order to save room, and to indicate to the student a more concise way of setting down his work; and the same will be observed in the following case.

CASE VII.

To subtract one fractional quantity from another.

Ex. 3. Here $\frac{12x}{7}-\frac{3x}{5}=\frac{60x}{35}-\frac{21x}{35}=\frac{39x}{35}=x+\frac{4x}{35}$ Ans.

Ex. 4. Here $15y-\frac{1+2y}{8}=\frac{120y-(1+2y)}{8}=\frac{118y-1}{8}$.

Ex. 5. Here $\left.\begin{array}{l} ax\times(b+c)=abx+acx \\ ax\times(b-c)=abx-acx \end{array}\right\}$ numerators.

Difference $\quad 2acx$

$(b-c)\times(b+c)=b^2-c^2$ common denominator.

Hence $\frac{2acx}{b^2-c^2}$ is the difference sought.

Ex. 6. Here $x-\frac{x-a}{c}-(x+\frac{x}{2b})=-\frac{x-a}{c}-\frac{x}{2b}=$ ...
$\frac{-2bx+2ba}{2bc}-\frac{cx}{2bc}=\frac{2ba-2bx-cx}{2bc}$ Ans.

Ex. 7. Here again we have $a+\frac{a-x}{a+x}-(a-\frac{a+x}{a-x}=$ ..
$\frac{a-x}{a+x}+\frac{a+x}{a-x}$.

Whence

$\left.\begin{array}{l} (a+x)\times(a+x)=a^2+2ax+x^2 \\ (a-x)\times(a-x)=a^2-2ax+x^2 \end{array}\right\}$ numerators.

Sum $=2a^2+2x^2$

$(a\times x)\times(a-x)=a^2-x^2$ common denominator.

Therefore $\frac{2a^2+2x^2}{a^2-x^2}$ is the difference required.

Ex. 8. This is the same, when properly arranged, as $ax-x+\frac{2x+7}{8}+\frac{5x-6}{21}=ax-x+\frac{42x+147}{168}+\frac{40x-48}{168}=$ $ax-x+\frac{82x+99}{168}=\frac{168ax-168x+82x+99}{168}=\ldots\ldots$ $\frac{168ax-86x+99}{168}=ax-\frac{86x-99}{168}$ the answer required.

Ex. 9. Here subtracting the first from the second, we have $x+\frac{11x-10}{15}-\frac{3x-5}{7}=\frac{26x-10}{15}-\frac{3x-5}{7}=\frac{182x-70}{105}-$ $\frac{45x-75}{105}=\frac{137x+5}{105}=x+\frac{32x+5}{105}$ the answer sought.

Ex. 10. First to find the difference of the fractions

$$\frac{a-x}{a(a+x)} \text{ and } \frac{a+x}{a(a-x)}$$

$$\left.\begin{array}{l} a(a-x)\times(a-x)=a^3-2a^2x+ax^2 \\ a(a+x)\times(a+x)=a^3+2a^2x+ax^2 \end{array}\right\} \text{numerators.}$$

Difference $=-4a^2x$

$a(a+x)\times a(a-x)=a^4-a^2x^2$ common denominator

Hence the second fraction subtracted from the first is

$$\frac{-4a^2x}{a^4-a^2x^2}=\frac{-4x}{a^2-x^2};$$

and consequently $a-\frac{4x}{a^2-x^2}$ is the difference sought.

CASE VIII.

To multiply fractional quantities together.

Ex. 4. Here* $\frac{\dot{3}x}{2}\times\frac{5x}{\dot{3}b}=\frac{5x^2}{2b}$ Ans.

Ex. 5. Here $\frac{\dot{2}x}{5}\times\frac{3x^2}{\dot{2}a}=\frac{3x^2}{5a}$ Ans.

Ex. 6. Here $\frac{2x}{3}\times\frac{4x^2}{7}\times\frac{a}{a+x}=\frac{8x^3a}{21a+21x}$ Ans.

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* These points are placed here to denote such factors as cancel each other.

Ex. 7. Here $\frac{\dot{2}x}{\dot{a}}\times\frac{\dot{3}a\dot{b}}{\dot{c}}\times\frac{5\dot{a}\dot{c}}{\dot{2}\dot{b}}=15ax$ Ans.

Ex. 8. By reducing these to improper fractions, they become $\frac{2a^2+bx}{a}\times\frac{3a^2x-b}{ax}=\frac{6a^4x+3a^2x^2b-2a^2b-b^2x}{a^2x}$.

It is, however, frequently more simple to multiply quantities of this kind together as in common multiplication, thus:

$$\begin{array}{r} 2a+\frac{bx}{a} \\ 3a-\frac{b}{ax} \\ \hline 6a^2+3bx \\ -\frac{2b}{x}-\frac{b^2}{a^2} \\ \hline 6a^2+3bx-\frac{2b}{a}-\frac{b^2}{a^2} \end{array}$$

the product, which is equivalent to the preceding fraction.

Ex. 9. Here $\frac{3x}{1}\times\frac{x+1}{2a}\times\frac{x-1}{a+b}=\frac{3x^3-3x}{2a^2+2ab}$ Ans.

Ex. 10. The third fraction $a+\frac{ax}{a-x}=\frac{a^2}{a-x}$.

So that we have $\frac{a^2-x^2}{a+b}\times\frac{a^2-b^2}{ax+x^2}\times\frac{a^2}{a-x}$.

Now observing what has been before stated relative to the factors $a^2-x^2=(a+x)(a-x)$, and $a^2-b^2=(a+b)(a-b)$, also $ax+x^2=x(a+x)$, our product may be written in the form

$$\frac{(a+x)(a-x)(a+b)(a-b)\times a^2}{(a+b)(a+x)x(a-x)}.$$

Which, by cancelling such factors in the numerator and denominator as are like, becomes

$$\frac{(a-b)a^2}{x}=\frac{a^3-a^2b}{x}, \text{ the answer.}$$

CASE IX.

To divide one fractional quantity by another.

Ex. 5. Here $\frac{7x}{5}\div\frac{3}{x}=\frac{7x}{5}\times\frac{x}{3}=\frac{7x^2}{15}$ Ans.

Ex. 6. Here $\frac{4x^2}{7}\div\frac{5x}{1}=\frac{4x^2}{7}\times\frac{1}{5x}=\frac{4x}{35}$ Ans.

Ex. 7. Here $\frac{x+1}{6}\times\frac{3}{2x}=\frac{x+1}{4x}$ Ans.

Ex. 8. Here $\frac{x}{1-x}\times\frac{5}{x}=\frac{5}{1-x}$ Ans.

Ex. 9. This may be written

$$\frac{(2a+x)x}{c^3-x^3}\times\frac{c-x}{x}=\frac{2a+x}{c^2+cx+x^2} \text{ Ans.}$$

Where it is to be observed that $c-x$ is a divisor of c^3-x^3 as stated in Case I.

Ex. 10. These two fractions are readily resolved into the following factors:

$$\frac{(x^2+b^2)(x^2-b^2)}{(x-b)(x-b)}\times\frac{x-b}{x(x+b)}=\frac{(x^2+b^2)(x^2-b^2)}{x(x^2-b^2)}=\ldots$$

$$=\frac{x^2+b^2}{x} \text{ Ans.}$$

INVOLUTION.

RULE I.

Ex. 1. $(2a^2)^3=2^3a^6=8a^6$ Ans.

Ex. 2. $(2a^2x)^4=2^4a^8x^4=16a^8x^4$ Ans.

Ex. 3. $(-\frac{2}{2}x^2y^3)^3=-\frac{2^3}{3^3}x^6y^9=-\frac{8}{27}x^6y^9$ Ans.

Ex. 4. $(\frac{3a^2x}{5b^2})^4=+\frac{3^4a^8x^4}{5^4b^8}=\frac{81^8ax^4}{625b^8}$.

Ex. 5. In the preceding page of the introduction we have

$$\begin{array}{l} (a+x)^3=a^3+3a^2x+3ax^2+x^3 \\ \qquad\qquad a+x \\ \hline \qquad\qquad a^4+3a^3x+3a^2x^2+ax^3 \\ \qquad\qquad\quad +\ a^3x+3a^2x^2+3ax^3+x^4 \\ \hline (a+x)^4=a^4+4a^3x+6a^2x^2+4ax^3+x^4 \quad \text{Ans.} \end{array}$$

Again, in the preceding page, by writing a for x, and y for a, we have

$$\begin{array}{l} (a-y)^3=a^3-3a^2y+3ay^2-y^3 \\ (a-y)^2=a^2-2ay+y^2 \\ \hline \qquad a^5-3a^4y+3a^3y^2-a^2y^3 \\ \qquad\quad -2a^4y+6a^3y^2-6a^2y^3+2ay^4 \\ \qquad\qquad +\ a^3y^2-3a^2y^3+3ay^4-y^5 \\ \hline (a-y)^5=a^5-5a^4y+10a^3y^2-10a^2y^3+5ay^4-y^5 \end{array}$$

INVOLUTION.

RULE II.

Ex. 3. Although the rule prescribes the finding the coefficients separately, it must not be understood as absolutely necessary, being merely stated in those terms for the sake of perspicuity: generally the whole operation is performed in one line, thus,

$$(a+x)^4=a^4+4a^3x+\frac{4.3}{2}a^2x^2+\frac{6.2}{3}ax^3+\frac{4.1}{4}x^4.$$

Or performing the divisions and multiplications
$(a+x)^4=a^4+4a^3x+6a^2x^2+4ax^3+x^4$.

And in the same manner, we have
$(a-x)^5=a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$.

Ex. 4. Here $(a+x)^6=$
$a^6+6a^5x+15a^4x^2+20a^3x^3+15a^2x^4+6ax^5+x^6$, and

$(a-y)^7=$
$-7a^6y+21a^5y^2-35a^4y^3+35a^3y^4-21a^2y^5+7ay^6-y$

Ex. 5. Here $(2+x)^5=$
$2^5+5.2^4x+10.2^3x^2+10.2^2x^3+5.2x^4+x^5$

Or establishing the powers of $2=$
$32+80x+80x^2+40x^3+10x^4+x^5$

In the other part of this example the quantity is a trinomial, which it is better to put under a binomial form, thus:

$$\left\{(a-bx)+c\right\}^3=(a-bx)^3+3(a-bx)^2c+3(a-bx)c^2+c^3$$

Then involving the several powers of $a-bx$ we have

$$\begin{aligned}(a-bx)^3&=a^3-3a^2bx+3ab^2x^2-b^3x^3\\ 3c(a-bx)^2&=3a^2c-6acbx+3cb^2x^2\\ 3c^2(a-bx)&=3c^2a-3c^2bx\\ c^3&=c^3\end{aligned}$$

Whence by addition $(a-bx+c)^3=$
$a^3+3a^2c+3c^2a+c^3-3a^2bx-6acbx-3c^2bx+3ab^2x^2+3cb^2x^2-b^3x^3$ Ans.

EVOLUTION.

CASE I.

To find the root of a simple quantity.

Ex. 3. Here $\sqrt{4a^2x^6}=\sqrt{4}\sqrt{a^2}\sqrt{x^6}=2ax^3$ Ans.

Ex. 4. Here $\sqrt[3]{-125a^3x^6}=\sqrt[3]{-125}\sqrt[3]{a^3}\sqrt[3]{x^6}=-5ax^2$.

Ex. 5. Here $\sqrt[4]{256a^4x^8}=\sqrt[4]{256}\sqrt[4]{a^4}\sqrt[4]{x^8}=4ax^2$.

Ex. 6. Here $\sqrt{\dfrac{4a^4}{9x^2y^2}}=\dfrac{\sqrt{4}\sqrt{a^4}}{\sqrt{9}\sqrt{x^2}\sqrt{y^2}}=\dfrac{2a^2}{3xy}$.

Ex. 7. Here $\sqrt[3]{\dfrac{8a^3}{125x^6}}=\dfrac{\sqrt[3]{8}\sqrt[3]{a^3}}{\sqrt[3]{125}\sqrt[3]{x^6}}=\dfrac{2a}{5x^2}$.

Ex. 8. Here $\sqrt[5]{\dfrac{-32a^5x^{10}}{243}}=\dfrac{\sqrt[5]{-32}\sqrt[5]{a^5}\sqrt[5]{x^{10}}}{\sqrt[5]{243}}=$ $\dfrac{-2ax^2}{3}$.

CASE II.

To extract the square root of a compound quantity.

Ex. 2. $a^4+4a^3x+6a^2x^2+4ax^3+x^4\,(a^2+2ax+x^2$ Ans.

$$\begin{array}{lll}
 & & a^4 \\
\hline
2a^2+2ax) & & 4a^3x+6a^2x^2 \\
\quad\;\; +2ax & & 4a^3x+4a^2x^2 \\
\hline
2a^2+4ax+x^2 & & \quad 2a^2x^2+4ax^3+x^4 \\
 & & \quad 2a^2x^2+4ax^3+x^4 \\
\hline
\end{array}$$

Ex. 3. $x^4-2x^3+\frac{3}{2}x^2-\frac{1}{2}x+\frac{1}{16}\,(x^2-x+\frac{1}{4}$ Ans.

$$\begin{array}{lll}
 & & x^4 \\
\hline
2x^2-x) & & -2x^3+\frac{3}{2}x^2 \\
\quad\;\; -x & & -2x^3+x^2 \\
\hline
2x^2-2x+\frac{1}{4}) & & \quad \frac{1}{2}x^2-\frac{1}{2}x+\frac{1}{16} \\
 & & \quad \frac{1}{2}x^2-\frac{1}{2}x+\frac{1}{16} \\
\hline
\end{array}$$

Ex. 4. $4x^6-4x^4+12x^3+x^2-6x+9\,(2x^3-x+3$ Ans.

$$\begin{array}{lll}
 & & 4x^6 \\
\hline
4x^3-x) & & -4x^4+12x^3+x^2 \\
\quad\;\; -x & & -4x^4+x^2 \\
\hline
4x^3-2x+3) & & \quad 12x^3-6x+9 \\
 & & \quad 12x^3-6x+9 \\
\hline
\end{array}$$

Ex. 5. $x^6+4x^5+10x^4+20x^3+25x^2+24x+16\,(x^3+2x^2+3x+4$

$$\begin{array}{lll}
 & & x^6 \\
\hline
2x^3+2x^2) & & 4x^5+10x^4 \\
\quad\;\; +2x^2 & & 4x^5+4x^4 \\
\hline
2x^3+4x^2+3x) & & 6x^4+20x^3+25x^2 \\
\qquad\qquad 3x & & 6x^4+12x^3+9x^2 \\
\hline
2x^3+4x^2+6x+4 & & \quad 8x^3+16x^2+24x+16 \\
 & & \quad 8x^3+16x^2+24x+16 \\
\hline
\end{array}$$

Ex. 6. $a^2+b\left(a+\frac{b}{2a}-\frac{b^2}{8a^3}+\text{\&c.}\right.$

$$a^2$$

$$2a+\frac{b}{2a}) \qquad +b$$

$$\frac{b}{2a} \qquad +b+\frac{b^2}{4a^2}$$

$$2a+\frac{b}{a}-\frac{b^2}{8a^3}) \quad -\frac{b^2}{4a^2}$$

$$-\frac{b^2}{4a^2}-\frac{b^3}{8a^4}+\frac{b^4}{64a^6}$$

$$\frac{b^3}{8a^4}-\frac{b^4}{64a^6}$$

Ex. 7. $1+1\left(1+\frac{1}{2}-\frac{1}{8}+\frac{1}{16}-\text{\&c.}\right.$

$$1$$

$$2+\frac{1}{2}) \qquad 1$$

$$\frac{1}{2} \qquad 1+\frac{1}{4}$$

$$2+1-\frac{1}{8}) \qquad -\frac{1}{4}$$

$$-\frac{1}{8} \qquad -\frac{1}{4}-\frac{1}{8}+\frac{1}{64}$$

$$2+1-\frac{1}{4} \qquad \frac{1}{8}-\frac{1}{64}$$

EVOLUTION.

CASE III.

To find any root of a compound quantity.

Ex. 3. $4a^2-12ax+9x^2\,(2a-3x$ Ans.

$$4a^2$$

$$4a)\ -12ax$$

$$4a^2-12ax+9x^2$$

Ex. 4. $a^2+2ab+2ac+b^2+2bc+c^2\,(a+b+c$

a^2

$2a)\ 2ab$

$(a+b)^2=a^2+2ab+b^2$

$2a)\quad 2ac$

$(a+b+c)^2=a^2+2ab+2ac+b^2+2bc+c^2$

Therefore $a+b+c$ is the root sought.

Ex. 5. $x^6-6x^5+15x^4-20x^3+15x^2-6x+1\,(x^2-2x+1$

x^6

$3x^4)\ -6x^5$

$(x^2-2x)^3=x^6-6x^5+12x^4-8x^3$

$3x^4)\quad 3x^4-12x^3$

$(x^2-2x+1)^3=x^6-6x^5+15x^4-20x^3+15x^2-6x+1$

Ans. x^2-2x+1

Ex. 6. $16a^4-96a^3x+216a^2x^2-216ax^3+81x^4\,(2a-3x$

$16a^4$

$32a^3)\ -96a^3x$

$(2a-3x)^4=16a^4-96a^3x+216a^2x^2-216ax^3+81x^4$

Ex. 7. $32x^5-80x^4+80x^3-40x^2+10x-1\,(2x-1$

$32x^5$

$80x^4)\ -80x^4$

$(2x-1)^5=32x^5-80x^4+80x^3-40x^2+10x-1$

IRRATIONAL QUANTITIES, OR SURDS.

CASE I.

To reduce a rational quantity to the form of a surd.

Ex. 3. Here $(5)^2=25$, therefore $\sqrt{25}$ Ans.

Ex. 4. Here $(-3x)^3=-27x^3$, therefore $\sqrt[3]{-27x^3}$ Ans.

Ex. 5. Here $(-2a)^4=16a^4$, therefore $-\sqrt[4]{16a^4}$, Ans.

Ex. 6. Here $(a^2)^5=a^{10}$, therefore $\sqrt[5]{a^{10}}$ Ans.

And $\sqrt{(a+\sqrt{b})^2}=a+b+2\sqrt{ab}. \therefore \sqrt{(a+b+2\sqrt{ab})}$, Ans.

$$\text{Again, } \left(\frac{\sqrt{a}}{2a}\right)^2=\frac{a}{4a^2}=\frac{1}{4a} \therefore \sqrt{\frac{1}{4a}}, \text{ Ans.}$$

$$\text{Also, } \left(\frac{a}{b\sqrt{a}}\right)^2=\frac{a^2}{b^2a}=\frac{a}{b^2} \therefore \sqrt{\frac{a}{b^2}}, \text{ Ans.}$$

Note to the above Case.

Ex. 1. Here $5\sqrt{6}=\sqrt{25}\times\sqrt{6}=\sqrt{150}$ Ans.

Ex. 2. Here $\frac{1}{5}\sqrt{5a}=\sqrt{\frac{1}{25}}\times\sqrt{5a}=\sqrt{\frac{a}{5}}$ Ans.

Ex. 3. Here $\frac{2a}{3}\sqrt[3]{\frac{9}{4a^2}}=\sqrt[3]{\frac{8a^3}{27}}\times\sqrt[3]{\frac{9}{4a^2}}=\sqrt[3]{\frac{72a^3}{108a^2}}=$ $\sqrt[3]{\frac{72a}{108}}=\sqrt[3]{\frac{2a}{3}}$ Ans.

CASE II.

To reduce quantities of different indices to others that shall have a given index.

Ex. 2. Here $\frac{1}{2}\div\frac{1}{6}=\frac{1}{2}\times\frac{6}{1}=3$, 1st index.

$\frac{1}{3}\div\frac{1}{6}=\frac{1}{3}\times\frac{6}{1}=2$, 2d index.

Hence $(5^3)^{\frac{1}{6}}$ and $(6^2)^{\frac{1}{6}}$ or $125^{\frac{1}{6}}$, and $36^{\frac{1}{6}}$ are the answers sought.

Ex. 3. Here $\frac{1}{2}\div\frac{1}{8}=\frac{1}{2}\times\frac{8}{1}=4$, 1st index.

$\frac{1}{4}\div\frac{1}{8}=\frac{1}{4}\times\frac{8}{1}=2$, 2d index.

Therefore $(2^4)^{\frac{1}{8}}$ and $(4^2)^{\frac{1}{8}}=16^{\frac{1}{8}}$ and $16^{\frac{1}{8}}$.

Whence $16^{\frac{1}{8}}$ and $16^{\frac{1}{8}}$ the answer sought.

Ex. 4. Here $\frac{2}{1}\div\frac{1}{4}=\frac{2}{1}\times\frac{4}{1}=8$ 1st index.

$\frac{1}{2}\div\frac{1}{4}=\frac{1}{2}\times\frac{4}{1}=2$ 2d index.

Therefore $(a^8)^{\frac{1}{4}}$ and $(a^2)^{\frac{1}{4}}$ are the quantities sought.

Ex. 5. Here $\frac{1}{2}\div\frac{1}{6}=\frac{1}{2}\times\frac{6}{1}=3$ 1st index.

$\frac{2}{3}\div\frac{1}{6}=\frac{2}{3}\times\frac{6}{1}=4$ 2d index.

Therefore $(a^3)^{\frac{1}{6}}$ and $(b^4)^{\frac{1}{6}}$ are the answers.

Note to the above Case.

Ex. 2. Here $\frac{1}{3}$ and $\frac{1}{4}=\frac{4}{12}$ and $\frac{3}{12}$

Hence $(4^4)^{\frac{1}{12}}$ and $(5^3)^{\frac{1}{12}}$ Ans.

Ex. 3. Here $\frac{1}{2}=\frac{3}{6}$ and $\frac{1}{3}=\frac{2}{6}$

Therefore $(a^3)^{\frac{1}{6}}$ and $(a^2)^{\frac{1}{6}}$ Ans.

Ex. 4. Here again $\frac{1}{3}=\frac{4}{12}$ and $\frac{1}{4}=\frac{3}{12}$

Therefore $(a^4)^{\frac{1}{12}}$ and $(b^3)^{\frac{1}{12}}$ Ans.

Ex. 5. Here $\frac{1}{n}$, $\frac{1}{m}$ reduced to a common denominator becomes $\frac{m}{mn}$ and $\frac{n}{mn}$.

Therefore $(a^m)^{\frac{1}{mn}}$ and $(b^n)^{\frac{1}{mn}}$ Ans.

CASE III.

To reduce surds to their most simple forms.

Ex. 3. Here $\sqrt{125}=\sqrt{(25\times5)}=5\sqrt{5}$ Ans.

Ex. 4. Here $\sqrt{294}=\sqrt{(49\times6)}=7\sqrt{6}$ Ans.

Ex. 5. Here $\sqrt[3]{56}=\sqrt[3]{(8\times7)}=2\sqrt[3]{7}$ Ans.

Ex. 6. Here $\sqrt[3]{192}=\sqrt[3]{(64\times3)}=4\sqrt[3]{3}$ Ans.

Ex. 7. Here $7\sqrt{80}=7\sqrt{(16\times5)}=28\sqrt{5}$ Ans.

Ex. 8. Here $9\sqrt[3]{81}=9\sqrt[3]{(27\times3)}=27\sqrt[3]{3}$ Ans.

Ex. 9. Here, reducing the radical, we have

$$\sqrt{\frac{5}{6}}=\sqrt{\frac{5\times6}{36}}=\frac{1}{6}\sqrt{30}$$

Therefore $\frac{3}{121}\sqrt{\frac{5}{6}}=\frac{3}{121}\times\frac{1}{6}\sqrt{30}=\frac{1}{242}\sqrt{30}$ Ans.

Ex. 10. $\sqrt[3]{\frac{3}{16}}=\sqrt[3]{\frac{3\times4}{64}}=\frac{1}{4}\sqrt[3]{12}$: hence $\frac{4}{7}\sqrt[3]{\frac{3}{16}}=$ $\frac{1}{7}\sqrt[3]{12}$, the answer.

Ex. 11. Here $\sqrt{98a^2x}=\sqrt{(49a^2\times2x)}=7a\sqrt{2x}$ Ans.

Ex. 12. Here $\sqrt{(x^3-a^2x^2)}=\sqrt{\{x^2(x-a^2)\}}=x\sqrt{(x-a^2)}$.

CASE IV.

To add surd quantities together.

Ex. 5. First $\sqrt{72}=\sqrt{(36\times2)}=6\sqrt{2}$
And $\sqrt{128}=\sqrt{(64\times2)}=8\sqrt{2}$

Ans. $14\sqrt{2}$

Ex. 6. Here $\sqrt{180}=\sqrt{(36\times5)}=6\sqrt{5}$
Also $\sqrt{405}=\sqrt{(81\times5)}=9\sqrt{5}$

Ans. $15\sqrt{5}$

Ex. 7. First $3\sqrt[3]{40}=3\sqrt[3]{(8\times5)}=6\sqrt[3]{5}$
And $\sqrt[3]{135}=\sqrt[3]{(27\times5)}=3\sqrt[3]{5}$

Ans. $9\sqrt[3]{5}$

Ex. 8. Here $4\sqrt[3]{54}=4\sqrt[3]{(27\times 2)}=12\sqrt[3]{2}$
And $5\sqrt[3]{128}=5\sqrt[3]{(64\times 2)}=20\sqrt[3]{2}$

$32\sqrt[3]{2}$

Ex. 9. Here $9\sqrt{243}=9\sqrt{(81\times 3)}=81\sqrt{3}$
And $10\sqrt{363}=10\sqrt{(121\times 3)}=110\sqrt{3}$

Ans. $191\sqrt{3}$

Ex. 10. By first reducing the fractional surds, we have

$$\sqrt{\frac{2}{3}}=\sqrt{\frac{6}{9}}=\frac{1}{3}\sqrt{6}$$

And $\sqrt{\frac{27}{50}}=\sqrt{\frac{54}{100}}=\frac{1}{10}\sqrt{(9\times 6)}=\frac{3}{10}\sqrt{6}$

Hence $3\sqrt{\frac{2}{3}}=3\times\frac{1}{3}\sqrt{6}=\sqrt{6}$

And $7\sqrt{\frac{27}{50}}=7\times\frac{3}{10}\sqrt{6}=\frac{21}{10}\sqrt{6}$

Ans. $=\frac{31}{10}\sqrt{6}$

Ex. 11. Here $\sqrt[3]{\frac{1}{4}}=\sqrt[3]{\frac{2}{8}}=\frac{1}{2}\sqrt[3]{2}$

And $\sqrt[3]{\frac{1}{32}}=\sqrt[3]{\frac{2}{64}}=\frac{1}{4}\sqrt[3]{2}$

Hence $12\sqrt[3]{\frac{1}{4}}+3\sqrt[3]{\frac{1}{32}}=6\sqrt[3]{2}+\frac{3}{4}\sqrt[3]{2}=6\frac{3}{4}\sqrt[3]{2}$ Ans.

Ex. 12. Here $\frac{1}{2}\sqrt{a^2b}=\frac{1}{2}\sqrt{(a^2\times b)}=\frac{1}{2}a\sqrt{b}$

And $\frac{1}{3}\sqrt{4bx^4}=\frac{1}{3}\sqrt{(4x^4\times b)}=\frac{2}{3}x^2\sqrt{b}$

Ans. $(\frac{1}{2}a+\frac{2}{3}x^2)\sqrt{b}$

CASE V.

To find the difference of surd quantities.

Ex. 1. Here $2\sqrt{50}=2\sqrt{(25\times2)}=10\sqrt{2}$
And $\sqrt{18}=\sqrt{(9\times2)}=3\sqrt{2}$

Difference $7\sqrt{2}$

Ex. 2. Here $\sqrt[3]{320}=\sqrt[3]{(64\times5)}=4\sqrt[3]{5}$
And $\sqrt[3]{40}=\sqrt[3]{(8\times5)}=2\sqrt[3]{5}$

Difference $2\sqrt[3]{5}$

Ex. 3. Here $\sqrt{\frac{3}{5}}=\sqrt{\frac{15}{25}}=\frac{1}{5}\sqrt{15}$
And $\sqrt{\frac{5}{27}}=\sqrt{\frac{15}{81}}=\frac{1}{9}\sqrt{15}$

Difference $\frac{4}{45}\sqrt{15}$

Ex. 4. Here $2\sqrt{\frac{1}{2}}=2\sqrt{\frac{2}{4}}=\sqrt{2}$
And $\sqrt{8}=\sqrt{(4\times2)}=2\sqrt{2}$

Difference $\sqrt{2}$

Ex. 5. Here $3\sqrt[3]{\frac{1}{3}}=3\sqrt[3]{\frac{9}{27}}=\sqrt[3]{9}$
And $\sqrt[3]{72}=\sqrt{(8\times9)}=2\sqrt[3]{9}$

Difference $\sqrt[3]{9}$

Ex. 6. Here $\sqrt[3]{\frac{2}{3}}=\sqrt[3]{\frac{18}{27}}=\frac{1}{3}\sqrt[3]{18}$
$\sqrt[3]{\frac{9}{32}}=\sqrt[3]{\frac{18}{64}}=\frac{1}{4}\sqrt[3]{18}$

Difference $\frac{1}{12}\sqrt[3]{18}$

Ex. 7. Here $\sqrt{80a^4x} = \sqrt{(16a^4 \times 5x)} = 4a^2\sqrt{5x}$
And $\sqrt{20a^2x^3} = \sqrt{(4a^2x^2 \times 5x)} = 2ax\sqrt{5x}$

Difference $(4a^2 \backsim 2ax)\sqrt{5x}$

Ex. 8. Here $8\sqrt[3]{a^3b} = 8\sqrt[3]{(a^3 \times b)} = 8a\sqrt[3]{b}$
And $2\sqrt[3]{a^6b} = 2\sqrt[3]{(a^6 \times b)} = 2a^2\sqrt[3]{b}$

Difference $(8a \backsim 2a^2)\sqrt[3]{b}$

CASE VI.

To multiply surd quantities together.

Ex. 5. Mult. $5\sqrt{8}$
By $3\sqrt{5}$

Product $= 15\sqrt{40} = 15\sqrt{(4 \times 10)} = 30\sqrt{10}$ Ans.

Ex. 6. Mult. $\sqrt[3]{18}$
By $5\sqrt[3]{4}$

Prod. $= 5\sqrt[3]{72} = 5\sqrt[3]{(8 \times 9)} = 10\sqrt[3]{9}$

Ex. 7. Mult. $\frac{1}{4}\sqrt{6}$
By $\frac{2}{15}\sqrt{9}$

Prod. $\frac{1}{30}\sqrt{54} = \frac{1}{30}\sqrt{(9 \times 6)} = \frac{1}{10}\sqrt{6}$

Ex. 8. Mult. $\frac{1}{2}\sqrt{18}$
By $5\sqrt{20}$

Prod. $= \frac{5}{2}\sqrt{360} = \frac{5}{2}\sqrt{(36 \times 10)} = 15\sqrt{10}$

Ex. 9. Mult. $2\sqrt{3}$
By $13\frac{1}{2}\sqrt{5}$

$27\sqrt{15}$ Answer.

Ex. 10. Here $72\frac{1}{4}a^{\frac{2}{3}} \times 120\frac{1}{2}a^{\frac{1}{4}} = \frac{289}{4} \times \frac{241}{2} \times a^{\frac{2}{3}} \times a^{\frac{1}{4}}$

$$=\frac{69649}{8}a^{\frac{8}{12}}\times a^{\frac{3}{12}}=8706\tfrac{1}{8}a^{\frac{11}{12}}\text{ Ans.}$$

Ex. 11. Mult. $4+2\sqrt{2}$
By $2-\sqrt{2}$

$8+4\sqrt{2}$
$-4\sqrt{2}-4$

$8-4=4$ the answer.

Ex. 12. Mult. $(a+b)^{\frac{1}{n}}=(a+b)^{\frac{m}{mn}}$

By $(a+b)^{\frac{1}{m}}=(a+b)^{\frac{n}{mn}}$

Product $=(a+b)^{\frac{m+n}{mn}}$

CASE VII.

To divide one surd quantity by another.

Ex. 5. Here $\frac{6\sqrt{54}}{3\sqrt{2}}=2\sqrt{27}=2\sqrt{(9\times3)}=6\sqrt{3}$ **Ans.**

Ex. 6. Here $\frac{4\sqrt[3]{72}}{2\sqrt[3]{18}}=2\sqrt[3]{4}$, which will not reduce lower.

Ex. 7. Here $5\frac{3}{4}\div\frac{2}{3}=\frac{23}{4}\times\frac{3}{2}=\frac{69}{8}$

And $\sqrt{\frac{1}{135}}\div\sqrt{\frac{1}{5}}=\sqrt{\frac{1}{27}}=\sqrt{\frac{3}{81}}=\frac{1}{9}\sqrt{3}$

Therefore $\frac{69}{8}\times\frac{1}{9}\sqrt{3}=\frac{69}{72}\sqrt{3}=\frac{23}{24}\sqrt{3}$ **Answer.**

Ex. 8. First $4\frac{5}{7}\div2\frac{2}{5}=\frac{33}{7}\times\frac{5}{12}=\frac{165}{84}$

And $\sqrt{\frac{2}{3}}\div\sqrt{\frac{3}{4}}=\sqrt{(\frac{2}{3}\times\frac{4}{3})}=\sqrt{\frac{8}{9}}=$

$$\sqrt{\frac{4\times2}{9}}=\frac{2}{3}\sqrt{2}$$

Hence $\frac{165}{84}\times\frac{2}{3}\sqrt{2}=\frac{55}{42}\sqrt{2}$ Answer.

Ex. 9. Here $4\frac{1}{2}\div2\frac{2}{3}=\frac{9}{2}\times\frac{3}{8}=\frac{27}{16}$

And $\sqrt{a}\div\sqrt[3]{ab}=a^{\frac{1}{2}}\div a^{\frac{1}{3}}b^{\frac{1}{3}}=\frac{a^{\frac{1}{6}}}{b^{\frac{1}{3}}}=\frac{a^{\frac{1}{6}}}{b^{\frac{2}{6}}}=$

$(\frac{a}{b^2})^{\frac{1}{6}}$; therefore $\frac{27}{16}(\frac{a}{b^2})^{\frac{1}{6}}$ the answer.

Ex. 10. Here $32\frac{2}{5}\div13\frac{3}{4}=\frac{162}{5}\times\frac{4}{55}=\frac{648}{275}$

And $\sqrt{a}\div\sqrt[3]{a}=a^{\frac{1}{2}}\div a^{\frac{1}{3}}=a^{\frac{3}{6}-\frac{2}{6}}=a^{\frac{1}{6}}$

Therefore $\frac{648}{275}a^{\frac{1}{6}}$ the answer required.

Ex. 11. Here $9\frac{3}{8}\div4\frac{9}{11}=\frac{75}{8}\times\frac{11}{53}=\frac{825}{424}$

And $a^{\frac{1}{n}}\div a^{\frac{1}{m}}=a^{\frac{m}{mn}}\div a^{\frac{n}{mn}}=a^{\frac{m-n}{mn}}$

Consequently $\frac{825}{424}a^{\frac{m-n}{mn}}$ the answer.

Ex. 12. Here $\frac{\sqrt{20}+\sqrt{12}}{\sqrt{5}+\sqrt{3}}=\sqrt{4}$, or 2, the answer.

Note to the above Case.

Ex. 3. Here $\frac{1}{a^2}=a^{-2}$ the answer.

Ex. 4. Here $a^{-\frac{1}{2}}=\frac{1}{a^{\frac{1}{2}}}$ the answer.

Ex. 5. Here $\frac{1}{a+x}=\frac{1}{(a+x)^1}=(a+x)^{-1}$. Ans.

Ex. 6. Here $a(a^2-x^2)^{-\frac{1}{3}}=\frac{a}{(a^2-x^2)^{\frac{1}{3}}}$ Ans.

CASE VIII.

To involve or raise surd quantities to any power.

Ex. 3. Here $(3\sqrt[3]{3})^2=(3\times 3^{\frac{1}{3}})^2=9.3^{\frac{2}{3}}=9\sqrt[3]{9}$ Ans.

Ex. 4. Here $(17\sqrt{21})^3=17^3\sqrt{(21^2\times 21)}=17^3\times 21$
$\sqrt{21}=103173\sqrt{21}$ Ans.

Ex. 5. Here $(\frac{1}{6}\sqrt{6})^4=(\frac{1}{6}\times 6^{\frac{1}{2}})^4=\frac{1}{6^4}\times 6^2=\frac{1}{6^2}$

Therefore $\frac{1}{36}$ the Ans.

Ex. 6. Multiply $3+2\sqrt{5}$
By $3+2\sqrt{5}$

$$\begin{array}{l} 9+6\sqrt{5} \\ \quad +6\sqrt{5}+4\sqrt{25} \\ \hline \end{array}$$

Square $=9+12\sqrt{5}+20=29+12\sqrt{5}$

Ex. 7. Multiply $\sqrt{x}+3\sqrt{y}$
By $\sqrt{x}+3\sqrt{y}$

$$\begin{array}{l} x+3\sqrt{xy} \\ \quad +3\sqrt{xy}+9y \\ \hline \end{array}$$

Square $=x+6\sqrt{xy}+9y$
Mult. by $\sqrt{x}+3\sqrt{y}$

$$\begin{array}{l} x\sqrt{x}+6x\sqrt{y}+9y\sqrt{x} \\ \quad +3x\sqrt{y}+18y\sqrt{x}+27y\sqrt{y} \\ \hline \end{array}$$

Cube $=x\sqrt{x}+9x\sqrt{y}+27y\sqrt{x}+27y\sqrt{y}$

Ex. 8. Here Mult. $\sqrt{3}-\sqrt{2}$
By $\sqrt{3}-\sqrt{2}$

$$\begin{array}{rl} & 3-\sqrt{6} \\ & \quad -\sqrt{6}+2 \\ \hline \text{Square} & = 5-2\sqrt{6} \\ \text{Square} & = 5-2\sqrt{6} \\ \hline & 25-10\sqrt{6} \\ & \quad -10\sqrt{6}+24 \\ \hline \end{array}$$

4th power $=49-20\sqrt{6}$ the answer.

CASE IX.

To find the roots of surd quantities.

Ex. 3. Here $\sqrt{10^3}=\sqrt{(10^2+10)}=10\sqrt{10}$ Ans.

Ex. 4. Here $\sqrt[3]{\frac{8}{27}a^4}=\sqrt[3]{(\frac{8}{27}a^3\times a)}=\frac{2}{3}a\sqrt[3]{a}$ Ans.

Ex. 5. Here $\sqrt[4]{\frac{16}{81}a^{\frac{2}{3}}}=\frac{2}{3}a^{\frac{2}{3}\times\frac{1}{4}}=\frac{2}{3}a^{\frac{1}{6}}$ Ans.

Ex. 6. Here $\frac{a}{3}\sqrt{\frac{a}{3}}=(\frac{a}{3})\times(\frac{a}{3})^{\frac{1}{2}}=(\frac{a}{3})^{\frac{3}{2}}$

Consequently $\sqrt[3]{(\frac{a}{3}\sqrt{\frac{a}{3}})}=\sqrt[3]{(\frac{a}{3})^{\frac{3}{2}}}=(\frac{a}{3})^{\frac{3}{6}}=(\frac{3a}{9})^{\frac{1}{2}}=\frac{1}{3}\sqrt{3a}$

Ex. 7. Here $x^2-4x\sqrt{a}+4a(x-2\sqrt{a}$ Ans.

$$\begin{array}{rl} & x^2 \\ \hline 2x-2\sqrt{a}) & -4x\sqrt{a}+4a \\ & -4x\sqrt{a}+4x \end{array}$$

Ex. 8. Here $a+2\sqrt{ab}+b(\sqrt{a}+\sqrt{b}$ Ans.

$$\begin{array}{rl} & a \\ \hline 2\sqrt{a}+\sqrt{b}) & 2\sqrt{ab}+b \\ & 2\sqrt{ab}+b \\ \hline \end{array}$$

CASE X.

To transform a binomial or residual surd into a general surd.

Ex. 4. First $(3-\sqrt{5})^2=9-6\sqrt{5}+5=14-6\sqrt{5}$, therefore $\sqrt{(14-6\sqrt{5})}$ Ans.

Ex. 5. Here $(\sqrt{2}-2\sqrt{6})^2=2-4\sqrt{12}+24=26-4\sqrt{12}$, therefore $\sqrt{(26-8\sqrt{3})}$ Ans.

Ex. 6. Here $(4-\sqrt{7})^2=16-8\sqrt{7}+7=23-8\sqrt{7}$
Hence $\sqrt{(23-8\sqrt{7})}$ Ans.

Ex. 7. In examples involving cube root radicals, it is useful to know the following form of the cube of a binomial: viz.

$$(a\pm b)^3=a^3\pm b^3+3ab(a\pm b)$$

Hence $(2\sqrt[3]{3}-3\sqrt[3]{9})^3=24-243-18\sqrt[3]{27}(2\sqrt[3]{3}-3\sqrt[3]{9})$ $=-219-54(2\sqrt[3]{3}-3\sqrt[3]{9})$

Consequently $\sqrt[3]{(-219-54(2\sqrt[3]{3}-3\sqrt[3]{9}))}$

is the general surd required.

CASE XI.

To extract the square root of a binomial surd.

Ex. 3. Here $\sqrt{[\frac{1}{2}a+\frac{1}{2}\sqrt{(a^2-b)}]}$* $=\sqrt{[3+\frac{1}{2}\sqrt{(36-20)}]}=\sqrt{(3+2)}$; and $\sqrt{[\frac{1}{2}a-\frac{1}{2}\sqrt{(a^2-b)}]}=\sqrt{[3-\frac{1}{2}\sqrt{(36-20)}]}=\sqrt{(3-2)}$

Hence $\sqrt{(3+2)}\pm\sqrt{(3-2)}=\sqrt{5}\pm1$ Ans.

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* It may here be observed, that b denotes the quantity under the second radical after its coefficient has been introduced. Thus in the present example, $b=20$, because $2\sqrt{5}=\sqrt{20}$.

Ex. 4. Here

$\sqrt{[\frac{1}{2}a+\frac{1}{2}\sqrt{(a^2-b)}]}=\sqrt{[\frac{23}{2}+\frac{1}{2}\sqrt{(529-448)}]}=\sqrt{(\frac{23}{2}+\frac{9}{2})}$

$\sqrt{[\frac{1}{2}a-\frac{1}{2}\sqrt{(a^2-b)}]}=\sqrt{[\frac{23}{2}-\frac{1}{2}\sqrt{(529-448)}]}=\sqrt{(\frac{23}{2}-\frac{9}{2})}$

Hence $\sqrt{(\frac{23}{2}+\frac{9}{2})}\pm\sqrt{(\frac{23}{2}-\frac{9}{2})}=4\pm\sqrt{7}$ **Ans.**

Ex. 5. Here

$\sqrt{[\frac{1}{2}a+\frac{1}{2}\sqrt{(a^2-b)}]}=\sqrt{[18+\frac{1}{2}\sqrt{(1296-1100)}]}=$ - - - -
$\sqrt{(18+7)}$; and

$\sqrt{[\frac{1}{2}a-\frac{1}{2}\sqrt{(a^2-b)}]}=\sqrt{[18-\frac{1}{2}\sqrt{(1296-1100)}]}=$ - - - -
$\sqrt{(18-7)}$

Hence $\sqrt{(18+7)}\pm\sqrt{(18-7)}=5\pm\sqrt{11}$ **Ans.**

Ex. 6. Here

$\sqrt{[\frac{1}{2}a+\frac{1}{2}\sqrt{(a^2-b)}]}=\sqrt{[\frac{33}{2}+\frac{1}{2}\sqrt{(1089-864)}]}=$ - - - -
$\sqrt{(\frac{33}{2}+\frac{15}{2})}$ and

$\sqrt{[\frac{1}{2}a-\frac{1}{2}\sqrt{(a^2-b)}]}=\sqrt{[\frac{33}{2}-\frac{1}{2}\sqrt{(1089-864)}]}=$ - - - -
$\sqrt{(\frac{33}{2}-\frac{15}{2})}$

Hence $\sqrt{(\frac{33}{2}+\frac{15}{2})}\pm\sqrt{(\frac{33}{2}-\frac{15}{2})}=\sqrt{24}\pm3$ **Ans.**

Ex. 7. Here $\sqrt{(\frac{1}{2}a+\frac{1}{2}\sqrt{(a^2+b)})}=\sqrt{(\frac{1}{2}+\frac{1}{2}\sqrt{(1+48)})}$
$=\sqrt{(\frac{1}{2}+\frac{7}{2})}=\sqrt{4}=2$;
And $\sqrt{(\frac{1}{2}a-\frac{1}{2}\sqrt{(a^2+b)})}=\sqrt{(\frac{1}{2}-\frac{1}{2}\sqrt{(1+48)})}=\sqrt{(\frac{1}{2}-\frac{7}{2})}$
$=\sqrt{-3}$.

Whence $\sqrt{(1+\sqrt{-48})}=2+\sqrt{-3}$. **Ans.**

Ex. 8. Here $\sqrt{(\frac{1}{2}a+\frac{1}{2}\sqrt{(a^2+b)})}=\sqrt{(\frac{3}{2}+\frac{1}{2}\sqrt{(9+16)})}$
$=\sqrt{(\frac{3}{2}+\frac{5}{2})}=\sqrt{4}=2$;
And $(\sqrt{\frac{1}{2}a-\frac{1}{2}\sqrt{(a^2+b)})}=\sqrt{(\frac{3}{2}-\frac{1}{2}\sqrt{(9+16)})}=\sqrt{(\frac{3}{2}-\frac{5}{2})}$
$=\sqrt{-1}$.

Whence $\sqrt{(3\pm\sqrt{-16})}=2\pm\sqrt{-1}$ **Ans.**

Ex. 9. Here $\sqrt{(\frac{1}{2}a+\frac{1}{2}\sqrt{(a^2+b)})}=\sqrt{(-\frac{1}{2}+\frac{1}{2}\sqrt{(1+8)})}$
$=\sqrt{(-\frac{1}{2}+\frac{3}{2})}=\sqrt{1}=1$;
And $\sqrt{(\frac{1}{2}a-\frac{1}{2}\sqrt{(a^2+b)})}=\sqrt{(-\frac{1}{2}-\frac{1}{2}\sqrt{(1+8)})}=$ - - -
$\sqrt{(-\frac{1}{2}-\frac{3}{2})}=\sqrt{-2}$.

Whence $\sqrt{(-1\pm\sqrt{-8})}=1\pm\sqrt{-2}$. **Ans.**

Ex. 10. Here $\sqrt{(\frac{1}{2}a+\frac{1}{2}\sqrt{(a^2-b)})}=\sqrt{(\frac{1}{2}a^2+\frac{1}{2}\sqrt{(a^4-4a^2x^2+4x^4)})}=\sqrt{(a^2-x^2)}$;

And $\sqrt{(\frac{1}{2}a-\frac{1}{2}\sqrt{(a^2-b)})}=\sqrt{(\frac{1}{2}a^2-\frac{1}{2}\sqrt{(a^4-4a^2x^2+4x^4)})}=x$.

Whence $\sqrt{(a^2+2x\sqrt{(a^2-x^2)})}=x+\sqrt{(a^2-x^2)}$ Ans.

Ex. 11. Let $-\sqrt{12}-\sqrt{24}$ be reduced to a general surd, and it becomes $-\sqrt{(+36+24\sqrt{2})}$. Hence $6+2\sqrt{2}=a$; and $+36+24\sqrt{2}=b$; therefore $\sqrt{(\frac{1}{2}a+\frac{1}{2}\sqrt{(a^2-b)})}=\sqrt{(3+\sqrt{2}+\frac{1}{2}\sqrt{(44+24\sqrt{2}-36-24\sqrt{2})})}=\sqrt{(3+\sqrt{2}+\frac{1}{2}\sqrt{8})}=\sqrt{(3+2\sqrt{2})}$*$=1+\sqrt{2}$.

Again, $\sqrt{(\frac{1}{2}a-\frac{1}{2}\sqrt{(a^2-b)})}=\sqrt{(3+\sqrt{2}-\frac{1}{2}\sqrt{(44+24\sqrt{2}-36-24\sqrt{2})})}=\sqrt{(3+\sqrt{2}-\frac{1}{2}\sqrt{8})}=\sqrt{(3+\sqrt{2}-\sqrt{2})}=\sqrt{3}$.

Whence $\sqrt{(6+2\sqrt{2}-\sqrt{12}-\sqrt{24})}=1+\sqrt{2}-\sqrt{3}$ Ans.

CASE XII.

To find any root of a binomial surd.

Ex. 1. We have $A^2-B^2=250=5\times5\times5\times2$; $\therefore Q\times5^3\times2=n^3$, whence $Q=4$, and $n=5\times2$. Then $\sqrt[3]{\{(A+B)\times\sqrt{Q}\}}$, or $\sqrt[3]{\left\{(68+\sqrt{(4374)}\times2)\right\}}=\sqrt[3]{268}+=r=7$, the nearest integer. $A\sqrt{Q}=68\sqrt{4}=136\sqrt{1}$, and the radical part $\sqrt{1}$ $=s$ and $\dfrac{r+\frac{n}{r}}{2s}=\dfrac{7+\frac{10}{7}}{2\sqrt{1}}=\dfrac{7+\frac{10}{7}}{2}=\dfrac{49+10}{14}=\dfrac{59}{14}=4=t$, the nearest integer. And $ts=4$, $\sqrt{(t^2s^2-n)}=\sqrt{(16-10)}=\sqrt{6}$, and $\sqrt[2\times3]{Q}=\sqrt[6]{Q}=\sqrt[6]{4}=\sqrt[3]{2}$; and so the root to be tried is $\dfrac{4-\sqrt{6}}{\sqrt[3]{2}}$ whose cube, upon trial, I find to be $68-\sqrt{4374}$.

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* The square root of this expression is found as in the foregoing examples.

Ex. 2. We have $A^2-B^2=175-121=54=2\times3\times3\times3$. $Q\times3^3\times2=n^3$; whence $n=3\times2=6$, and $Q=4$ (because $3^3\times2\times4=3^3\times2^3=n^3$). Then $\sqrt[3]{[(A+B)\times\sqrt{Q}]}=\sqrt[3]{[(13+11)\times2]}=\sqrt[3]{48}+=r=4$. $A\sqrt{Q}=\sqrt{175}\times\sqrt{4}=2\sqrt{175}=10\sqrt{7}$, and the radical part $\sqrt{7}=s$, and $\dfrac{r+\frac{n}{r}}{2s}=\dfrac{4+\frac{6}{4}}{2\sqrt{7}}$ $=t=1$ in the nearest integer. And $ts=\sqrt{7}$, $\sqrt{[(t^2s^2-n)]}=\sqrt{(7-6)}=\sqrt{1}=1$. $\sqrt[2c]{Q}=\sqrt[6]{4}=\sqrt[3]{2}$; whence $\dfrac{ts+\sqrt{[(t^2s^2-n)]}}{\sqrt[2c]{Q}}=\dfrac{\sqrt{7}+1}{\sqrt[3]{2}}$ the Ans.

Ex. 3. We have $A^2-B^2=1$; $Q\times1=n^3$, whence $Q=1$, and $n=1$. Then $\sqrt{[(A+B)\times\sqrt{Q}]}=\sqrt[3]{10}+=r=2$ nearly. $A\sqrt{Q}=2\sqrt{7}$, and hence the radical part $\sqrt{7}=s$, and $\dfrac{r+\frac{n}{r}}{2s}=\dfrac{2+\frac{1}{2}}{2\sqrt{7}}=\dfrac{5}{10}=t=\dfrac{1}{2}$.* And $ts=\dfrac{\sqrt{7}}{2}$, $\sqrt{(t^2s^2-n)}=$ . . $\sqrt{(\frac{7}{4}-1)}=\sqrt{\frac{3}{4}}=\dfrac{\sqrt{3}}{2}$. Therefore $\dfrac{ts+\sqrt{(t^2s^2-n)}}{\sqrt[2c]{Q}}=$ . . . $\dfrac{\sqrt{7}+\sqrt{3}}{2}$ Ans.

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* The rule in the Introduction, which was *first* given by Newton in the UNIVERSAL ARITHMETIC, fails when $t=\frac{1}{2}$ exactly, as it would be in the above example, if the calculation was performed accurately. I proposed this particular example at the Mathematical Club: "Required to know, if the cube root of $2\sqrt{7}+3\sqrt{3}$ can be found by the *rule* given by Newton, page 139, Universal Arithmetic, for extracting any root of a binomial surd; and, if not, to show where that rule fails, and what alteration is to be made in it, so as to obtain the root?" Robert Adrain, LL. D. Professor of Mathematics, &c. Columbia College, has since ably investigated the subject, and *he* found the *rule* not only to fail in this, but in a great variety of other examples; and has also discovered the *rule* to be defective. See "The Exercises of the Mathematical Club of the City of New-York."

Ex. 4. We have $A^2-B^2=3$; $Q\times 3=n^5$; hence $Q=3^4$, and $n=3$. Then $\sqrt[5]{\{(A+B)+\sqrt{Q}\}}=\sqrt[5]{\{\sqrt{5046}+\sqrt{(5043)}\times\sqrt{81}\}}=\sqrt[5]{(71+71)\times 9\}}=r=4$ nearly. $A\sqrt{Q}=\sqrt{5046}\times\sqrt{81}=261\sqrt{6}$, and hence the radical part $\sqrt{6}=s$

and $\dfrac{r+\frac{n}{r}}{2s}=\dfrac{4+\frac{3}{4}}{2\sqrt{6}}=1=t$ in the nearest integer, and $ts=\sqrt{6}$, $\sqrt{(t^2s^2-n)}=\sqrt{(6-3)}=\sqrt{3}$ and $\sqrt[2c]{Q}=\sqrt[10]{81}=\sqrt[5]{9}$.

And therefore $\dfrac{ts+\sqrt{(t^2s^2-n)}}{\sqrt[2c]{Q}}=\dfrac{\sqrt{6}+\sqrt{3}}{\sqrt[5]{9}}$ the Ans.

Ex. 5. Here $a=45$, and $b=1682$; whence $\sqrt[3]{(a^2-b)}=\sqrt[3]{(2025-1682)}=7$, and $n^3-3(\sqrt[3]{(a^2-b)})n=90$, or $n^3-21n=90$, whence it readily appears from inspection, that $n=6$. Whence

$\sqrt[3]{(45+\sqrt{1682})}=\frac{6}{2}+\frac{1}{2}\sqrt{(36-4\sqrt{343})}=3+\frac{1}{2}\sqrt{8}=3+\sqrt{2}$

$\sqrt[3]{(45-\sqrt{1682})}=\frac{6}{2}-\frac{1}{2}\sqrt{(36-4\sqrt{343})}=3-\frac{1}{2}\sqrt{8}=3-\sqrt{2}$.

Ex. 6. Here $a=9$, and $b=80$; whence $\sqrt[3]{(a^2-b)}=\sqrt[3]{(81-80)}=1$, and $n^3-3(\sqrt[3]{(a^2-b)})n=18$, or $n^3-3n=18$. Whence it readily appears, by making trials of the divisors of 18, which are 1, 2, 3, &c. for n, that $n=3$; whence

$\sqrt[3]{(9+\sqrt{80})}=\frac{3}{2}+\frac{1}{2}\sqrt{\{9-4\sqrt[3]{(81-80)}\}}=\frac{3}{2}+\frac{1}{2}\sqrt{5}$,

$\sqrt[3]{(9-\sqrt{80})}=\frac{3}{2}-\frac{1}{2}\sqrt{\{9-4\sqrt[3]{(81-80)}\}}=\frac{3}{2}-\frac{1}{2}\sqrt{5}$.

Ex. 7. Here $a=20$, and $b=32368$; whence $\sqrt[3]{(a^2+b)}=\sqrt[3]{(400+32368)}$ and $n^3-3(\sqrt[3]{(a^2+b)})n=40$, or $n^3-96n=40$; when it readily appears, by inspection, that $n=10$; whence

$\sqrt[3]{\{20+\sqrt{32368}\}}=\frac{10}{2}+\frac{1}{2}\sqrt{\{100-4\sqrt[3]{(400+32368)}\}}=5+\sqrt{-7}$,

$\sqrt[3]{\{20-\sqrt{32368}\}}=\frac{10}{2}-\frac{1}{2}\sqrt{\{100-4\sqrt[3]{(400+32368)}\}}=5-\sqrt{-7}$.

Ex. 8. Here $a=35$, and $b=28566$; whence $\sqrt[3]{(a^2+b)}=\sqrt[3]{(1225+28566)}=31$, and $n^3-3(\sqrt[3]{(a^2+b)}n)=70$, or $n^3-93n=70$; when it readily appears, by inspection, that $n=10$; whence

$\sqrt[3]{(35+\sqrt{28566})}=\frac{10}{2}+\frac{1}{2}\sqrt{(100-4\times 31)}=5+\sqrt{-6}$,

$\sqrt[3]{(35-\sqrt{28566})}=\frac{10}{2}-\frac{1}{2}\sqrt{(100-4\times 31)}=5-\sqrt{-6}$.

Ex. 9. Here $81+\sqrt{-2700}=27\times(3+\sqrt{-\frac{100}{27}})$; then the cube root of $3+\sqrt{-\frac{100}{27}}$, can be now more easily found. We have $a=3$, and $b=\frac{100}{27}$; whence $\sqrt[3]{(9+\frac{100}{27})}=\frac{7}{3}$, and $n^3-3\left\{\sqrt[3]{(a^2+b)}\right\}n=n^3-7n=6$; when it readily appears that $n=-2$; whence $\sqrt[3]{(3+\sqrt{-\frac{100}{27}})}=-1+\frac{1}{2}\sqrt{(4-4\times\frac{7}{3})}=-1+\frac{2}{3}\sqrt{-3}$, we shall have by multiplying by 3, (which is the cube root of 27,) $-3+2\sqrt{-3}$; and in like manner, $\sqrt[3]{(3-\sqrt{-\frac{100}{27}})}=-1-\frac{2}{3}\sqrt{-3}$, which, multiplied by 3, as before, becomes $-3-2\sqrt{-3}$.

CASE XIII.

To find such a multiplier, or multipliers, as will make any binomial surd rational.

Ex. 5. Given surd $\sqrt{5}-\sqrt{x}$
Multiplier $\sqrt{5}+\sqrt{x}$

Product $5-x$ as required.

Ex. 6. Given surd $\sqrt{a}+\sqrt{b}$
Multiplier $\sqrt{a}-\sqrt{b}$

Product $a-b$ as required.

Ex. 7. Given surd $a+\sqrt{b}$
Multiplier $a-\sqrt{b}$

Product a^2-b the answer.

Ex. 8. Given surd $1-\sqrt[3]{2a}$
Square of the terms $=1+\sqrt[3]{4a^2}$
Product with sign changed $=+\sqrt[3]{2a}$
Therefore $1+\sqrt[3]{2a}+\sqrt[3]{4a^2}=$ multiplier.
Mult. by $1-\sqrt[3]{2a}$

$$\begin{array}{l} 1+\sqrt[3]{2a}+\sqrt[3]{4a^2} \\ \quad -\sqrt[3]{2a}-\sqrt[3]{4a^2}-\sqrt[3]{8a^3} \end{array}$$

Product $=1-\sqrt[3]{8a^3}=1-2a$ as required.

Ex. 9. Given surd $\sqrt[3]{3}-\frac{1}{2}\sqrt[3]{2}$
Square of the terms $\sqrt[3]{9}+\frac{1}{4}\sqrt[3]{4}$
Product with sign changed $+\frac{1}{2}\sqrt[3]{6}$
Whence $\sqrt[3]{9}+\frac{1}{2}\sqrt[3]{6}+\frac{1}{4}\sqrt{4}$ is the multiplier required.

Ex. 10. Given surd $a^{\frac{3}{4}}+b^{\frac{3}{4}}$

Here $m=\frac{3}{4}$ and $n=3$, and $a^{n-m}-a^{n-2m}b^{m}+a^{n-3m}b^{2m}$ &c. $=a^{3-\frac{3}{4}}-a^{3-\frac{6}{4}}b^{\frac{3}{4}}+a^{3-\frac{9}{4}}b^{\frac{3}{2}}-a^{3-3}b^{\frac{9}{4}}$ $=\sqrt[4]{a^9}-\sqrt[4]{(a^6b^3)}+\sqrt[4]{a^3b^6}-\sqrt[4]{b^9}$.*

CASE XIV.

To reduce a fraction whose denominator is either a simple or compound surd, to another that shall have a rational denominator.

Ex. 4. Here $\frac{\sqrt{6}}{\sqrt{7}+\sqrt{3}}=\frac{\sqrt{6}}{\sqrt{7}+\sqrt{3}}\times\frac{\sqrt{7}-\sqrt{3}}{\sqrt{7}-\sqrt{3}}=$ - - - $\frac{\sqrt{42}-\sqrt{18}}{4}$ Ans.

Ex. 5. Here $\frac{x}{3+\sqrt{x}}=\frac{x}{3+\sqrt{x}}\times\frac{3-\sqrt{x}}{3-\sqrt{x}}=\frac{3x-x\sqrt{x}}{9-x}$. Ans.

Ex. 6. Here $\frac{a-\sqrt{b}}{a+\sqrt{b}}=\frac{a-\sqrt{b}}{a+\sqrt{b}}\times\frac{a-\sqrt{b}}{a-\sqrt{b}}=\frac{a^2+b-2a\sqrt{b}}{a^2-b}$ which is the answer sought.

Ex. 7. Here, by the preceding rule, the multiplier for the denominator is $\sqrt[3]{49}+\sqrt[3]{35}+\sqrt[3]{25}$.

Whence $\frac{10}{\sqrt[3]{7}-\sqrt[3]{5}}=\frac{10}{\sqrt[3]{7}-\sqrt[3]{5}}\times\frac{\sqrt[3]{49}+\sqrt[3]{35}+\sqrt[3]{25}}{\sqrt[3]{49}+\sqrt[3]{35}+\sqrt[3]{25}}=$ $\frac{10(\sqrt[3]{49}+\sqrt[3]{35}+\sqrt[3]{25})}{7-6}=5(\sqrt[3]{49}+\sqrt[3]{35}+\sqrt[3]{25})=$ the answer required.

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* Here, by taking $n=4$, and putting a^3 and b^3 for a and b in the formule, we shall obtain the same result.

Ex. 8. Here $\frac{\sqrt[3]{3}}{\sqrt[3]{9}+\sqrt[3]{10}}=\frac{\sqrt[3]{3}}{\sqrt[3]{9}+\sqrt[3]{10}}\times$ . . . .
$\frac{\sqrt[3]{81}-\sqrt[3]{90}+\sqrt[3]{100}}{\sqrt[3]{81}-\sqrt[3]{90}+\sqrt[3]{100}}=\frac{\sqrt[3]{3}(\sqrt[3]{81}-\sqrt[3]{90}+\sqrt[3]{100})}{9+10}=$ . . . .
$\frac{3\sqrt[3]{9}-3\sqrt[3]{10}+\sqrt[3]{300}}{19}$ Ans.

Ex. 9. Here $\frac{4}{\sqrt[4]{4}+\sqrt[4]{5}}=\frac{4}{\sqrt[4]{4}+\sqrt[4]{5}}\times\frac{\sqrt[4]{4}-\sqrt[4]{5}}{\sqrt[4]{4}-\sqrt[4]{5}}=$ . . .
$\frac{4(\sqrt[4]{4}-\sqrt[4]{5})}{\sqrt[2]{4}-\sqrt[2]{5}}=\frac{4(\sqrt[4]{4}-\sqrt[4]{5})}{2-\sqrt[2]{5}}\times\frac{2+\sqrt[2]{5}}{2+\sqrt[2]{5}}=$ . . . .
$\frac{4(\sqrt[4]{4}-\sqrt[4]{5})(2+\sqrt{5})}{4-5}=4(\sqrt[4]{5}-\sqrt[4]{4})(2+\sqrt{5})=4(-\sqrt{10}$
$-2\sqrt{2}+(2+\sqrt{5})\times\sqrt[4]{5})$ Ans.

ARITHMETICAL PROPORTION AND PROGRESSION.

Ex. 3. Here the formula $S=(a+l)\times\frac{n}{2}$ becomes $s=(1+1000)\times 500=500500$ Ans.

Ex. 4. Here the formula $s=\{2a+(n-1)d\}\frac{n}{2}$ becomes $\{2+100\times 2\}\times\frac{101}{2}=202\times\frac{101}{2}=101^2=10201$.

Ex. 5. Here $s=(a+l)\frac{n}{2}=(1+24)\times 12=300$ Ans.

Ex. 6. Here the formula $l=a+(n-1)d$, becomes $l=2+(365-1)2=2+728=730$ Ans.

Ex. 7. Here $s=\{2a-(n-1)d\}\frac{n}{2}=\{20-20\times\frac{1}{3}\}\times\frac{21}{2}=(20-\frac{20}{3})\times\frac{21}{2}=\frac{40}{3}\times\frac{21}{2}=140$ Ans.

Ex. 8. Here the first term is $1+1=2$, and the last $100+100=200$, the number of terms 100.

Therefore $(a+l)\frac{n}{2}=(2+200)50=10100$ yards, or 5 miles 1300 yards, the Answer.

GEOMETRICAL PROPORTION AND PROGRESSION.

Ex. 3. Here the 1st term $a=1$, the ratio $r=2$, the number of terms $n=20$;

Whence the formula $s=a(\frac{r^n \backsim 1}{r \backsim 1})$ becomes $1\times\frac{2^{20}-1}{2-1}=$ $2^{20}-1=1048575$ Ans.

Ex. 4. Here $a=1$, $r=\frac{1}{2}$, and $n=8$, whence $s=a(\frac{r^n \backsim 1}{r \backsim 1})$ $=1\times\frac{1-(\frac{1}{2})^8}{1-\frac{1}{2}}=\frac{2^8-1}{2^8\times\frac{1}{2}}=\frac{2^8-1}{2^7}=\frac{255}{128}=1\frac{127}{128}$ the answer, or sum required.

Ex. 5. Here $a=1$, $r=\frac{1}{3}$, and $n=10$; whence $s=a$ $(\frac{r^n \backsim 1}{r \backsim 1})=1\times\frac{1-(\frac{1}{3})^{10}}{1-\frac{1}{3}}=\frac{3^{10}-1}{3^{10}\times\frac{2}{3}}=\frac{3^{10}-1}{3^9\times2}=\frac{59048}{39366}=1\frac{9841}{19683}$

Ex. 6. Here $a=1$, $r=2$, $n=32$. Whence $s=(a\frac{r^n \backsim 1}{r \backsim 1})$ $=1\times\frac{2^{32}-1}{2-1}=2^{32}-1=4294967295$ farthings $=4473924l.$ $5s.$ $3\frac{3}{4}d.$

EQUATIONS.

RESOLUTIONS OF SIMPLE EQUATIONS.

CASE 1.

1. Here $2x+3=x+17$, by transposing gives $2x-x=17-3=14$. Whence $x=14$ Ans.

2. Here $5x-9=4x+7$, by transposing, gives $5x-4x=7+9=16$. Whence $x=16$ Ans.

3. Here $x+9-2=4$, by transposing, gives $x=4+2-9=-3$ Ans.

4. Here $9x-8=8x-5$, by transposing, gives $9x-8x=8-5=3$. Whence $x=3$ Ans.

5. Here $7x+8-3=6x+4$, by transposing, gives $7x-6x=4+3-8=-1$. Whence $x=-1$ Ans.

CASE II.

1. Here $16x+2=34$, by transposing, gives $16x=34-2=32$, and by division $x=\frac{32}{16}=2$. Ans.

2. Here $4x-8=-3x+13$, by transposing, gives $4x+3x=13+8$, or $7x=21$, and, by division, $x=\frac{21}{7}=3$ Ans.

3. Here $10x-19=7x+17$, by transposition, $10x-7x=17+19$, or $3x=36$, and by division, $x=\frac{36}{3}=12$ Ans.

4. Here $8x-3+9=-7x+9+27$, by transposition $8x+7x=27+9-9+3$, or $15x=30$, and, by division, $x=\frac{30}{15}=2$ the answer.

5. Here $3ax-3ab=12d$, by transposition, $3ax=12d+3ab$, and, by division, $x=\frac{12d+3ab}{3a}=b+\frac{4d}{a}$ Ans.

CASE III.

1. Here $\frac{3x}{2}=\frac{x}{4}+24$; then multiplying by 4, (which is a multiple of 4 and 2,) $6x=x+96$, or $6x-x=96$, or $5x=96$, and, by division, $x=\frac{96}{5}=19\frac{1}{5}$ Ans.

2. Here $\frac{x}{3}+\frac{x}{5}+\frac{x}{2}=62$; then, multiplying by 3, we have $x+\frac{3x}{5}+\frac{3x}{2}=186$, and multiplying by 5, gives $5x+3x+\frac{15x}{2}=930$; and this, multiplied by 2, gives $10x+6x+15x=1860$, or $31x=1860$, and, by division $x=\frac{1860}{31}=60$ Ans.

3. Here $\frac{x-3}{2}+\frac{x}{3}=20-\frac{x+19}{2}$; then, multiplying by 6, (which is a multiple of 2, 3, and 2,) $3x-9+2x=120-3x-57$, and, by transposition and division, $x=\frac{72}{8}=9$ Ans.

4. Here $\frac{x+1}{2}+\frac{x+2}{3}=16-\frac{x+3}{4}$; then, multiplying by 12, we get $6x+6+4x+8=192-3x-9$, or $13x=169$; and by division $x=\frac{169}{13}=13$ Ans.

5. Here $\frac{x+a}{b}+\frac{x}{c}=\frac{2x}{a}+\frac{a+b}{d}$; then, multiplying by b, $x+a+\frac{bx}{c}=\frac{2bx}{a}+\frac{ab+b^2}{d}$, mult. by c, $cx+ca+bx=\frac{2bcx}{a}+\frac{abc+cb^2}{d}$ and, by multiplying by a and d successively, we have $adcx+a^2cd+abdx=2bcdx+a^2bc+acb^2$, or, by transposition $adcx+abdx-2bcdx=a^2bc+acb^2-a^2cd$, and by division $x=\frac{a^2bc+acb^2-a^2cd}{adc+abd-2bcd}$ Ans.

CASE IV.

1. Here $2\sqrt{x}+3=9$; then, $2\sqrt{x}=9-3=6$ by transposition, and $4x=36$ by squaring, or $x=\frac{36}{4}=9$ Ans.

2. Here $\sqrt{(x+1)}-2=3$; then $\sqrt{(x+1)}=3+2=5$, or $x+1=25$ by squaring, and consequently $x=25-1=24$ Ans.

3. Here $\sqrt[3]{(3x+4)}+3=6$; then $\sqrt[3]{(3x+4)}=6-3=3$, or $3x+4=27$ by cubing, and $3x=27-4=23$, or $x=\frac{23}{3}=7\frac{2}{3}$ the answer.

4. Here $\sqrt{(4+x)}=4-\sqrt{x}$; then $4+x=16-8\sqrt{x}+x$ by squaring, and, by trans. $8\sqrt{x}=12$, or $2\sqrt{x}=3$; and $4x=9$ by squaring, or $x=\frac{9}{4}=2\frac{1}{4}$ Ans.

5. Here $\sqrt{(4a^2+x^2)}=\sqrt[4]{(4b^4+x^4)}$; then, by squaring $4a^2+x^2=\sqrt{(4b^4+x^4)}$, and squaring again, $16a^4+8a^2x^2+x^4=4b^4+x^4$, by transposition and division $x^2=\frac{b^4-4a^4}{2a^2}$ and consequently $x=\sqrt{(\frac{b^4-4a^4}{2a^2})}$ Ans.

CASE V.

1. Here $9x^2-6=30$; then $9x^2=30+6=36$, or $x^2=\frac{36}{9}=4$, and consequently $x=\sqrt{4}=2$ Ans.

2. Here $x^3+9=36$; then $x^3=36-9=27$, or $x=\sqrt[3]{27}=3$ the answer.

3. Here $x^2+x+\frac{1}{4}=\frac{81}{4}$; then, by extracting the square root of both sides $x+\frac{1}{2}=\frac{9}{2}=4\frac{1}{2}$, or $x=4\frac{1}{2}-\frac{1}{2}=4$ Ans.

4. Here $x^2+ax+\frac{a^2}{4}=b^2$; then, by extracting the square root of both sides, $x+\frac{a}{2}=b$, or $x=b-\frac{a}{2}$ Ans.

5. Here $x^2+14x+49=121$, then, by extracting the square root of both sides of the equation $x+7=11$, or $x=11-7=4$ Ans.

CASE VI.

1. Here $\frac{3}{4}x : a :: 5bc : cd$; then $\frac{3}{4}x \times cd = a \times 5bc$, or $3cdx = 20abc$, consequently, $x=\frac{20abc}{3cd}=\frac{20ab}{3d}$ Ans.

2. Here $10-x : \frac{2}{3}x :: 3 : 1$, then $10-x=2x$ by mult. ext. and means, and $3x=10$, or $x=\frac{10}{3}=3\frac{1}{3}$ Ans.

3. Here $8+8x : 4x :: 8 : 2$; then $16+16x=32x$ by mult. ext. and means, and $32x-16x=16$, or $16x=16$; therefore $x=\frac{16}{16}=1$ Ans.

4. Here $x : 6-x :: 2 : 4$; then $4x=12-2x$ by mult. ext. and means, and $4x+2x=12$, or $6x=12$; therefore $x=\frac{12}{6}=2$ Ans.

5. Here $4x : a :: 9\sqrt{x} : 9$; then $36x=9a\sqrt{x}$ by mult. ext. and means, or $4x=a\sqrt{x}$, and by squaring $16x^2=a^2x$; therefore, by division, $x=\frac{a^2}{16}$ Ans.

EXAMPLES FOR PRACTICE.

Ex. 1. Here $3x-2+24=31$, by transposing
Gives $3x=31+2-24=9$
Whence $x=\frac{9}{3}=3$ Ans.

Ex. 2. Here $4-9y=14-11y$, or
$11y-9y=14-4$, or $2y=10$,
Whence $y=\frac{10}{2}=5$ Ans.

Ex. 3. Here $x+18=3x-5$, or $3x-x=18+5$
or $2x=23$, whence $x=11\frac{1}{2}$.

Ex. 4. Here $x+\frac{x}{2}+\frac{x}{3}=11$

Mult. by 6. $6x+3x+2x=66$, or $11x=66$,
Whence $x=6$

Ex. 5. Multiply the given equation by 2, and we have
$4x-x+2=10x-4$; whence $10x+x-4x=4+2$,
or $7x=6$, whence $x=\frac{6}{7}$

Ex. 6. Mult. $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}=\frac{7}{10}$ by 60, gives
$30x+20x-15x=42$, or $35x=42$,
or $x=\frac{42}{35}=\frac{6}{5}=1\frac{1}{5}$ Ans.

Ex. 7. Mult. $\frac{x+3}{2}+\frac{x}{3}=4-\frac{x-5}{4}$, by 12, gives
$6x+18+4x=48-3x+15$, or
$13x=45$, whence $x=\frac{45}{13}=3\frac{6}{13}$ Ans.

Ex. 8. Here $2+\sqrt{3x}=\sqrt{(4+5x)}$ being squared,
Gives $4+4\sqrt{3x}+3x=4+5x$
Whence $4\sqrt{3x}=5x-3x=2x$
Squaring $48x=4x^2$, or dividing by $4x$, we have
$x=12$.

Ex. 9. Here $x+a=\frac{x^2}{x+a}$, or $x^2+2ax+a^2=x^2$

Whence $2ax=-a^2$, or $2x=-a$, or $x=-\frac{a}{2}$

Ex. 10. Here $\sqrt{x}+\sqrt{(a+x)}=\frac{2a}{\sqrt{(a+x)}}$, mult. by $\sqrt{(a+x)}$

Gives $\sqrt{(ax+x^2)}+a+x=2a$, or $\sqrt{(ax+x^2)}=a-x$;

Hence by squaring $ax+x^2=a^2-2ax+x^2$,

Conseq. $3ax=a^2$, or $3x=a$, or $x=\frac{a}{3}$ Ans.

Ex. 11. The equation $\frac{ax-b}{4}+\frac{a}{3}=\frac{bx}{2}-\frac{bx-a}{3}$

Mult. by 12 gives $3ax-3b+4a=6bx-4bx+4a$

Whence $3ax-2bx=3b$, or $x(3a-2b)=3b$

$$\text{Consequently } x=\frac{3b}{3a-2b}$$

Ex. 12. Here $\sqrt{(a^2+x^2)}=\sqrt[4]{(b^4+x^4)}$, by squaring

Gives $a^2+x^2=\sqrt{(b^4+x^4)}$, squaring again

Gives $a^4+2a^2x^2+x^4=b^4+x^4$

Whence $2a^2x^2=b^4-a^4$, or $x=\sqrt{\frac{b^4-a^4}{2a^2}}$

Ex. 13. Here $\sqrt{(a+x)}+\sqrt{(a-x)}=\sqrt{ax}$, by squaring

Gives $2a+2\sqrt{(a^2-x^2)}=ax$,

Whence $\sqrt{(a^2-x^2)}=\frac{ax-2a}{2}$; squaring again,

$$a^2-x^2=\frac{a^2x^2-4a^2x+4a^2}{4}$$

Therefore $-4x^2=a^2x^2-4a^2x$, or dividing by x;

$-4x=a^2x-4a^2$, or

$$x=\frac{4a^2}{a^2+4}.$$

Ex. 14. Here $\frac{a}{1+x}+\frac{a}{1-x}=b$, becomes by reduction

$$\frac{a-ax+a+ax}{1-x^2}=b, \text{ or } \frac{2a}{1-x^2}=b$$

Whence $2a=b-bx^2$, or $bx^2=b-2a$

$$\text{Therefore } x=\sqrt{\frac{b-2a}{b}}$$

Ex. 15. By squaring $a+x=\sqrt{\{a^2+x\sqrt{(b^2+x^2)}\}}$.

We have $a^2+2ax+x^2=a^2+x\sqrt{(b^2+x^2)}$

Transposing $2ax+x^2=x\sqrt{(b^2+x^2)}$

Divide by x, $2a+x=\sqrt{(b^2+x^2)}$

By squaring $4a^2+4ax+x^2=b^2+x^2$

Whence $4ax=b^2-4a^2$, or $x=\frac{b^2}{4a}-a$

The answer required.

Ex. 16. Multiplying the given equation by 2, we have

$\sqrt{(x^2+3a^2)}-\sqrt{(x^2-3a^2)}=2x\sqrt{a}$.

By squaring $2x^2-2\sqrt{(x^4-9a^4)}=4ax^2$, or

$-\sqrt{(x^4-9a^4)}=2ax^2-x^2$

Squaring again $x^4-9a^4=4a^2x^4-4ax^4+x^4$, or

$(4a-4a^2)x^4=9a^4$, or $x=\sqrt[4]{\frac{9a^3}{4-4a}}$

Ex. 17. Here $\sqrt{(a+x)}+\sqrt{(a-x)}=b$, by squaring

$2a+2\sqrt{(a^2-x^2)}=b^2$, or $\sqrt{(a^2-x^2)}=\frac{1}{2}b^2-a$,

Whence $a^2-x^2=\frac{1}{4}b^4-b^2a+a^2$

And $x=\sqrt{(b^2a-\frac{1}{4}b^4)}=\frac{b}{2}\sqrt{(4a-b^2)}$

Ex. 18. Given equation $\sqrt[3]{(a+x)}+\sqrt[3]{(a-x)}=b$

By cubing both sides after the form of Ex. 7. Case I. Surds, we have

$2a+3\sqrt[3]{(a^2-x^2)}\{\sqrt[3]{(a+x)}+\sqrt[3]{(a-x)}\}=b^3$,

But since $\sqrt[3]{(a+x)}+\sqrt[3]{(a-x)}=b$, this becomes

$2a+3\sqrt{(a^2-x^2)}\times b=b^3$, or

$3\sqrt[3]{(a^2-x^2)}=\frac{b^3-2a}{b}$, or

$\sqrt[3]{(a^2-x^2)}=\frac{b^3-2a}{3b}$ whence

$a^2-x^2=\frac{(b^3-2a)^3}{27b^3}=(\frac{b^3-2a}{3b})^3$

Therefore $x=\sqrt{\{a^2-(\frac{b^3-2a}{3b})^3\}}$

Ex. 19. Here $\sqrt{a}+\sqrt{x}=\sqrt{ax}$, which divided by $\sqrt{x}$

Gives $\frac{\sqrt{a}}{\sqrt{x}}+1=\sqrt{a}$, or $\frac{\sqrt{a}}{\sqrt{x}}=-1+\sqrt{a}$,

or $\sqrt{a}-1$.

Whence $\sqrt{(a-1)}\sqrt{x}=\sqrt{a}$, or $\sqrt{x}=\frac{\sqrt{a}}{\sqrt{(a-1)}}$

Therefore $x=\frac{a}{(\sqrt{a}-1)^2}$

Ex. 20. Here $\sqrt{\frac{x+1}{x-1}}+\sqrt{\frac{x-1}{x+1}}=a$, by squaring

Gives $\frac{x+1}{x-1}+\frac{x-1}{x+1}+2\sqrt{\frac{x^2-1}{x^2-1}}=a^2$

Or $\frac{2x^2+2}{x^2-1}+2=a^2$; or by multiplying

$2x^2+2+2x^2-2=x^2a^2-a^2$,

Whence $4x^2-a^2x^2=-a^2$, or $(a^2-4)x^2=a^2$

Therefore $x=\frac{a}{\sqrt{(a^2-4)}}$ the answer.

Ex. 21. This equation transposed, is

$\sqrt{(a^2+ax)}+\sqrt{(a^2-ax)}=a$, squaring

$2a^2+2\sqrt{(a^4-a^2x^2)}=a^2$; whence

$\sqrt{(a^4-a^2x^2)}=-\frac{1}{2}a^2$; squaring again,

$a^4-a^2x^2=\frac{1}{4}a^4$, or $a^2x^2=\frac{3}{4}a^4$,

Whence $x=\sqrt{\frac{3}{4}a^2}=\frac{a}{2}\sqrt{3}$

Ex. 22. Here $\sqrt{(a^2-x^2)}+x\sqrt{(a^2-1)}=a^2\sqrt{(1-x^2)}$; then, by transposition, $\sqrt{(a^2-x^2)}=a^2\sqrt{(1-x^2)}-x\sqrt{(a^2-1)}$; squaring, $a^2-x^2=a^4-a^4x^2-2xa^2\sqrt{(1-x^2)(a^2-1)}+a^2x^2-x^2$. Again, by transposition, and division, $2x\sqrt{(1-x^2)(a^2-1)}=a^2-a^2x^2+x^2-1=(1-x^2)(a^2-1)$; by squaring, $4x^2(1-x^2)(a^2-1)=(1-x^2)^2(a^2-1)^2$, and $4x^2=(1-x^2)(a^2-1)$, dividing the whole by $(1-x^2)(a^2-1)$: therefore by actually multiplying, $4x^2=a^2-a^2x^2+x^2-1$, or $3x^2+a^2x^2=a^2-1$; whence $x^2=\frac{a^2-1}{a^2+3}$; and consequently, $x=\sqrt{(\frac{a^2-1}{a^2+3})}$ Ans.

Ex. 23. Here $\sqrt{(x+a)}=c-\sqrt{(x+b)}$; squaring, $x+a=c^2-2c\sqrt{(x+b)}+x+b$, and transposing

$$2c\sqrt{(x+b)}=c^2+b-a, \text{ or } \sqrt{(x+a)}=\frac{c^2+b-a}{2c}$$

$$\text{whence } x+b=\left(\frac{c^2+b-x}{2c}\right)^2, \text{ and } x=\left(\frac{c^2+b-a}{2c}\right)^2-b;$$

the answer.

Ex. 24. Here $\sqrt{\frac{b}{a+x}}+\sqrt{\frac{c}{a-x}}=\sqrt{\frac{4bc}{a^2-x^2}}$, by squaring,

$$\text{Gives } \frac{b}{a+x}+\frac{c}{a-x}+\sqrt{\frac{4bc}{a^2-x^2}}=\sqrt{\frac{4bc}{a^2-x^2}}$$

$$\text{Whence } \frac{b}{a+x}+\frac{c}{a-x}=0; \text{ or}$$

$$\frac{ab-bx+ac+cx}{a^2-x^2}=0,$$

Therefore $ab-bx+ac+cx=0$,
And $bx-cx=ab+ac$

$$\text{Consequently } x=\frac{ab+ac}{b-c}=a\left(\frac{b+c}{b-c}\right)$$

The resolution of simple equations, containing two unknown quantities.

RULE I.

EXAMPLES FOR PRACTICE.

1. Here $4x+y=34$, and $4y+x=16$, then from 1st $x=\frac{34-y}{4}$, and from 2d, $x=16-4y$; therefore, by equality,

$\frac{34-y}{4}=16-4y$, and, consequently $34-y=64-16y$, or $16y-y=64-34$, or $15y=30$; whence $y=\frac{30}{15}=2$, and $x=16-4y=16-8=8$.

2. Here $2x+3y=16$, and $3x-2y=11$; then, from 1st $x=\frac{16-3y}{2}$ and from 2d, $x=\frac{11+2y}{3}$ therefore by equality, $\frac{11+2y}{3}=\frac{16-3y}{2}$ and, conseq. $22+4y=48-9y$, or $13y=48-22=26$; whence $y=\frac{26}{13}=2$, and $x=\frac{11+4}{3}=\frac{15}{3}=5$.

3. Here $\frac{2x}{5}+\frac{3y}{4}=\frac{9}{20}$, and $\frac{3x}{4}+\frac{2y}{5}=\frac{61}{120}$; then, mult. the 1st by 20, and the 2d by 20; we shall have $8x+15y=9$, and $15x+8y=10\frac{1}{6}$; whence, from the former equation, $x=\frac{9-15y}{8}$, and from the latter $x=\frac{10\frac{1}{6}-8y}{15}$; therefore, by equality, $\frac{10\frac{1}{6}-8y}{15}=\frac{9-15y}{8}$ or $81\frac{1}{3}-64y=135-225y$, hence $161y=53\frac{2}{3}$, or $y=53\frac{2}{3}\div161=\frac{1}{3}$; also $x=\frac{9-15y}{8}=\frac{9-5}{8}=\frac{1}{2}$.

4. Here $\frac{1}{2}x+2y=a$, and $\frac{1}{2}x-2y=b$; then, from the 1st $x=2a-4y$; and from the 2d, $x=2b+4y$, hence $2b+4y=2a-4y$, or $8y=2a-2b$; whence $y=\frac{1}{4}a-\frac{1}{4}b$, and $x=2b+4y=2b+a-b=a+b$.

5. Here $\frac{x}{2}+\frac{y}{3}=8$, $\frac{x}{3}-\frac{y}{2}=1$; then from the first $x=16-\frac{2y}{3}$, and from the 2d, $x=3+\frac{3y}{2}$, hence $3+\frac{3y}{2}=16-\frac{2y}{3}$, or $18+9y=96-4y$; whence $y=\frac{78}{13}=6$, and $x=16-\frac{2y}{3}=16-\frac{12}{3}=12$.

6. Here $\frac{x}{2}+\frac{y}{3}=9$, and $x:y::4:3$; then, from the 1st $x=18-\frac{2y}{3}$, and from the 2d $x=\frac{4y}{3}$, hence, $\frac{4y}{3}=18-\frac{2y}{3}$, or $4y=54-2y$; whence $y=\frac{54}{6}=9$; and $x=\frac{4y}{3}=\frac{36}{3}=12$.

7. Here $x+y=80$, and $\frac{2x}{3}=\frac{3y}{4}$: then, from the 1st, $x=80-y$, and from the 2d $x=\frac{9y}{8}$, hence $\frac{9y}{8}=80-y$, or $9y+8y=640$, whence $y=\frac{640}{17}=37\frac{11}{17}$, and $x=80-y=80-37\frac{11}{17}=42\frac{6}{17}$.

8. Here $y-6=\frac{x}{2}$, and $x=y+6$; then, from the 1st, $x=2y-12$, and therefore, by equality, $2y-12=y+6$; whence $y=18$, and $x=y+6=18+6=24$.

RULE II.

EXAMPLES FOR PRACTICE.

1. Here $\frac{x}{7}+7y=99$, and $\frac{y}{7}+7x=51$; then, from the 1st $x=693-49y$; which value, being substituted for x, in the second, gives $\frac{y}{7}+7(693-49y)=51$, or $2400y=33600$, whence $y=14$, and $x=693-49y=693-686=7$.

2. Here $\frac{x}{2}-12=\frac{y}{4}+8$, and $\frac{x+y}{5}+\frac{x}{3}-8=\frac{2y-x}{4}+27$; then, from the second, $x=\frac{18y+2100}{47}$; which value, being substituted for x, in the first gives $\frac{9y+1050}{47}-12=\frac{y}{4}+8$, or, clearing of fractions and transposing, $11y=440$, or $y=\frac{440}{11}=40$; whence $x=\frac{18y+2100}{47}=\frac{720+2100}{47}=60$.

3. Here $x+y=s$ and $x^2-y^2=d$; then, from the first, $x=s-y$, or $x^2=s^2-2sy+y^2$; which value being substituted for x, in the 2d, gives $s^2-2sy+y^2-y^2=d$, or $y=\frac{s^2-d}{2s}$ and $x=s-y=s-\frac{s^2-d}{2s}=\frac{s^2+d}{2s}$.

4. Here $5x-3y=150$, and $10x+15y=825$; then, from the 1st, $x=\frac{150+3y}{5}$; which value being substituted for x, in the second, we shall have $\frac{1500+30y}{5}+15y=825$, or $21y=525$, whence $y=\frac{525}{21}=25$, and $x=\frac{150+3y}{5}=\frac{150+75}{5}=45$.

5. Here $x+y=16$, and $x:y::3:1$; then, from the 2d, $x=3y$ which value of x, being substituted in the 1st, we shall have $3y+y=16$, or $y=4$, and consequently $x=12$.

6. Here $x+\frac{y}{2}=12$, and $y+\frac{x}{2}=9$; then from the 2d, $x=18-2y$; which value, being substituted in the 1st we shall have $18-2y+\frac{y}{2}=12$; hence $y=4$, and $x=10$.

7. Here $x:y::3:2$, and $x^2-y^2=20$; then, from the 1st, $x=\frac{3y}{2}$, or $x^2=\frac{9y^2}{4}$ which value being substituted in the 2d, gives $\frac{9y^2}{4}-y^2=20$, or $y^2=16$, hence $y=\sqrt{16}=4$ and $x=\frac{3y}{2}=\frac{12}{2}=6$.

8. Here $\frac{x}{2}-12=\frac{y}{4}+13$, and $\frac{x+y}{5}+\frac{x}{3}+16=\frac{2x-y}{4}+27$; then, from the 2d, $x=\frac{660-27y}{2}$ which value, being substituted in the 1st, we shall have $\frac{660-27y}{4}=\frac{y}{4}+25$; hence, by transposing and reducing $28y=560$, or $y=\frac{560}{28}=20$, and $x=\frac{660-27y}{2}=\frac{660-540}{2}=60$.

RULE III.

EXAMPLES FOR PRACTICE.

Ex. 1. Here $\frac{x+8}{4}+6y=21$, and $\frac{y+6}{3}=23-5x$; then, from the 1st, $x+24y=76$, and from the 2d, $y+15x=63$. Multiply the 1st by 15, and we have $15x+360y=1140$, then subtract the 2d equation from this, and we shall have $359y=1077$, or $y=\frac{1077}{359}=3$, and from the first, $x=76-24y=76-72=4$.

Ex. 2. Here $3x+7y=79$, and $2y=9+\frac{x}{2}$; then, from the 2d, $4y-x=18$. Multiply this equation by 3, and we have $12y-3x=54$, to which add the 1st, and we shall have $19y=133$, or $y=\frac{133}{19}=7$, and, hence from the 2d equation $x=4y-18=28-18=10$.

Ex. 3. Here $30x+40y=270$, and $50x+30y=340$; then, multiplying the 1st by 5, and the 3d by 3, we shall have $150x+200y=1350$, and $150x+90y=1020$, then, by subtracting the latter from the former, gives $110y=330$, or $y=\frac{330}{110}=3$, and from the 1st $x=\frac{270-40y}{30}=\frac{270-120}{30}=5$.

Ex. 4. Here $3x-3y=2x+2y$, and $x+y:xy::3:5$; then, from the 1st $x=5y$, and from the 2d, $5x+5y=3xy$, or by substituting the value of x, found from the 1st, we shall have $25y+5y=15y^2$, and dividing by y and transposing $y=\frac{30}{15}=2$, and $x=5y=10$.

Ex. 5. Here $x^2y+xy^2=30$, and $x^3+y^3=35$; then, adding 3 times the 1st equation to the 2d, gives $x^3+3x^2y+3xy^2+y^3=125$, and by extracting the cube root $x+y=5$, and, from the 1st $(x+y)xy=30$, hence by substitution $5xy=30$, or $xy=6$, and $x^2+2xy+y^2=25$, from which subtract 4 times the last equation, and we shall have $x^2-2xy+y^2=1$ and by evol. $x-y=1$, adding this equation to $x+y=5$, and we shall have $2x=6$, or $x=3$, whence $y=2$.

Ex. 6. Here $\frac{3x-5y}{2}=\frac{2x+y}{5}-3$, and $-8+\frac{x-2y}{4}=\frac{x}{2}+\frac{y}{3}$; then, from the first, $27y-11x=30$, and from the 2d, $9x-2y=96$. Multiplying the 1st of these equations by 2, and the second by 27, we shall have $54y-22x=60$, and $243x-54y=2592$; then, by adding the two last equations together, we have $221x=2652$, or $x=\frac{2652}{221}=12$, and $9x-2y=96$, hence $2y=9x-96=108-96=12$, hence $y=\frac{12}{2}=6$.

Ex. 7. In the first of the two equations $(x+y):a::(x-y):b$, and $x^2-y^2=c$;

$$\therefore b(x+y)=a(x-y), \text{ or } (b+a)y=(a-b)x,$$

$$\text{Whence } y=\frac{a-b}{a+b}x, \text{ and } y^2=\frac{(a-b)^2}{(a+b)^2}x^2$$

Substitute this for y^2 in the 2d equation, and we have $x^2-\frac{(a-b)^2}{(a+b)^2}x^2=c$, cr

$$x^2(a+b)^2-(a-b)^2x^2=c(a+b)^2, \text{ or}$$

$$x^2\left\{(a+b)^2-(a-b)^2\right\}=c(a+b)^2, \text{ or}$$

$$x^2(4ab)=c(a+b)^2; \text{ whence } x^2=\frac{c(a+b)^2}{4ab} \text{ ro}$$

$$x=\frac{a+b}{2}\sqrt{\frac{c}{ab}},\ y^2=x^2-c=\frac{c(a+b)^2}{4ab}-c=\frac{c(a-b)^2}{4ab},$$

$$\text{or } y=\frac{a-b}{2}\sqrt{\frac{c}{ab}} \text{ the answer}$$

Ex. 8. Given $\left.\begin{array}{l}ax+by=c\\ dx+ey=f\end{array}\right\}$

Mult. 1st by d, $dax+dby=dc$
Mult. 2d by a, $dax+aey=af$
By subt. $dby-aey=dc-af$

Whence $y=\dfrac{dc-af}{db-ae}=\dfrac{af-dc}{ae-db}$.

Mult. 1st by e, $eax+eby=ec$
Mult. 2d by b, $bdx+eby=bf$
By subtraction $eax-bdx=ec-bf$

Whence $x=\dfrac{ec-bf}{ea-bd}=\dfrac{bf-ce}{bd-ea}$

Ex. 9. Given $x+y=a$, and $x^2-y^2=b$.
Here divide $x^2-y^2=b$

by $x+y=a$; and we have $x-y=\dfrac{b}{a}$

Whence by add. $2x=a+\dfrac{b}{a}$, or $x=\dfrac{a^2+b}{2a}$

And by subt. $2y=a-\dfrac{b}{a}$, or $y=\dfrac{a^2-b}{2a}$

Ex. 10. Here $\left.\begin{array}{l}x^2+xy=a\\ y^2+xy=b\end{array}\right\}$

By add. $x^2+2xy+y^2=a+b$, or
$(x+y)^2=a+b$, or $x+y=\sqrt{(a+b)}$.

Now the two proposed equations may be put under the form

$x(x+y)=a$, or $x\sqrt{(a+b)}=a$
$y(x+y)=b$, or $y\sqrt{(a+b)}=b$

Whence by division, $x=\dfrac{a}{\sqrt{(a+b)}}$; and $y=\dfrac{b}{\sqrt{(a+b)}}$

The Resolution of Simple Equations, containing three or more unknown Quantities.

PRACTICAL EXAMPLES.

Ex. 3. Given $\left.\begin{array}{l} x+y+z=53 \\ x+2y+3z=105 \\ x+3y+4z=134 \end{array}\right\}$

Subtract 1st from 2d, $\left.\begin{array}{l} y+2z=52 \\ y+z=29 \end{array}\right\}$
Subtract 2d from 3d,

Now, subtracting the latter from the preceding one, we have $z=23$.

Also from the last $y=29-z=29-23=6$,

And from the first $x=53-y-z=53-29=24$,

That is $x=24$, $y=6$, and $z=23$.

Ex. 4. Given $\left.\begin{array}{l} x+\frac{1}{2}y+\frac{1}{3}z=32 \\ \frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=15 \\ \frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=12 \end{array}\right\}$

Multiplying the first by 6, and the second and third by 60, we have

$$\left.\begin{array}{l} 6x+3y+2z=192 \\ 20x+15y+12z=900 \\ 15x+12y+10z=720 \end{array}\right\}$$

Again multiply the first by 10, the second by 3, and the third by 4, give

$$\begin{array}{l} 60x+30y+20z=1920 \\ 60x+45y+36z=2700 \\ 60x+48y+40z=2880 \end{array}$$

Subtracting now the first of these, from the 2d, and the second from the 3d, we have

$$\left.\begin{array}{l} 15y+16z=780 \\ 3y+4z=180 \end{array}\right\}$$

Mult. the latter by 5, $15y+20z=900$
Subtract - - - - $15y+16z=780$

$4z=120$, or $z=30$

But $y=\frac{180-4z}{3}=20$, and $x=\frac{192-3y-2z}{6}=12$

herefore $x=12$, $y=20$, and $z=30$.

Ex. 5. Given

$$\begin{aligned} 7x+5y+2z&=\ 79 \\ 8x+7y+9z&=122 \\ x+4y+5z&=\ 55 \end{aligned}$$

Multiply the first by 8, the second by 7, and the third by 56, and we have

$$\left.\begin{aligned} 56x+\ 40y+\ 16z&=\ 632 \\ 56x+\ 49y+\ 63z&=\ 854 \\ 56x+224y+280z&=3080 \end{aligned}\right\}$$

Subtract the first from the second, and the second from the third, and we obtain

$$\left.\begin{aligned} 9y+\ 47z&=\ 222 \\ 175y+217z&=2226 \end{aligned}\right\}$$

Multiply the first of these by 175, and the second by 9, and we have

$$\begin{aligned} 1575y+8225z&=38850 \\ 1575y+1953z&=20034 \end{aligned}$$

By subtraction, $6272z=18816$, or $z=3$

But $y=\dfrac{222-47z}{9}=9$, and $x=\dfrac{55-4y-5z}{1}=4$

That is, $x=4$, $y=9$, and $z=3$.

Ex. 6. Given

$$\left.\begin{aligned} x+y&=a \\ x+z&=b \\ y+z&=c \end{aligned}\right\}$$

By addition, $2x+2y+2z=a+b+c$, or
$x+\ x+\ z=\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}c$;

From which, subtracting each of the three given equations respectively, we have

$$\begin{aligned} z&=-\tfrac{1}{2}a+\tfrac{1}{2}b+\tfrac{1}{2}c \\ y&=\ \ \tfrac{1}{2}a-\tfrac{1}{2}b+\tfrac{1}{2}c \\ x&=\ \ \tfrac{1}{2}a+\tfrac{1}{2}b-\tfrac{1}{2}c \end{aligned}$$

The values sought.

Ex. 7. Given $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=62$, $\frac{x}{3}+\frac{y}{4}+\frac{z}{5}=47$, and

$$\frac{x}{4}+\frac{y}{5}+\frac{z}{6}=38.$$

By clearing of fractions, these become

$$6x+4y+3z=744,\ 20x+15y+12z=2820,$$
$$\text{and } 15x+12y+10z=2280,$$

Multiplying the first by 20, the second by 6, and the third by 8, we shall have

$$120x+80y+60z=14880$$
$$120x+90y+72z=16920$$
$$120x+96y+80z=18240.$$

And, subtracting the first of these equations from the second, and the second from the third, there will arise $10y+12z=2040$, and $6y+8z=1320$.

Or, multiplying the first of these two equations by 3, and the second by 5, we shall have

$$30y+36z=6120, \text{ and } 30y+40z=6600.$$

Whence, by subtracting the former of these from the latter, we have $4z=480$, or $z=120$.

And, consequently, by substitution and reduction, $x=24$, and $y=60$.

Ex. 8. Given $z+y=x+100$, $y-2x=2z-100$, and $z+100=3x+3y$; then, by transposing, $z+y-x=100$, $y-2x-2z=-100$, and $z-3x-3y=-100$. And, by adding twice the first to the second, and subtracting the third from the first, we shall have $3y-4x=100$, and $2x+4y=200$; multiplying the second of these two equations by 2, and adding the result to the first, we have $11y=500$, or $y=45\frac{5}{11}$. And, consequently, by substitution and reduction, $x=9\frac{1}{11}$, and $z=63\frac{7}{11}$.

Ex. 9. Given $x+y+z=7$, $2x-3=y+3z$, and $5x+5z=3y+19$; then, by adding the first to the second, and 3 times the first to the third, and transposing the terms of the second and third, we shall have $3x-2z=10$, and $8x+8z=40$. Or, dividing the latter of these two equations by 4, we have $2x+2z=10$, and, adding this to the former, $5x=20$, or $x=4$, and, consequently, $z=1$, and $y=2$.

Ex. 10. Given $3x+5y-4z=25$, $5x-2y+3z=46$, and $3y+5z-x=62$; then, adding 3 times the third to the first, and 5 times the third to the second, we shall have

$$14y+11z=211, \text{ and } 13y+28z=356.$$

Or, multiplying the first of these two equations by 13, and the second by 14, there will result,

$$182y+143z=2743, \text{ and } 182y+392z=4984.$$

And, subtracting the former from the latter,

$$249z=2241, \text{ or } z=\frac{2241}{249}=9,$$

And, consequently, by substitution and reduction, $y=8$, and $x=7$.

Ex. 11. Given $x+y+z=13$, $x+y+u=17$, $x+z+u=18$, and $y+z+u=21$; then, assume $S=x+y+z+u$, and the above equations will be transformed into the following ones,

$$S-u=13, S-z=17, S-y=18, \text{ and } S-x=21.$$

Add all these equations together, and we have $4S-x-y-z-u=69$, that is, $4S-(x+y+z+u)=69$.

But $x+y+z+u=S$, therefore, $4S-S=69$, or $3S=69$, and $S=23$, and, consequently, by substituting for S its value in the four transformed equations, we shall have

$$23-u=13, 23-z=17, 23-y=18, \text{ and } 23-x=21.$$

And, consequently, $x=2$, $y=5$, $z=6$, and $u=10$.

SIMPLE EQUATIONS.

Ex. 1. Let one of the parts $=x$,

Then the other will be $=15-x$,

And by the question $15-x=\frac{3}{4}x$, or

$60-4x=3x$, or $7x=60$; whence $x=\frac{60}{7}=8\frac{4}{7}$, the one part; and $15-x=6\frac{3}{7}$, the other.

Ex. 2. Let the value of the purse be x; then the money $=7x$; and by the question $7x+x=20$, or $8x=20$, or $x=\frac{20}{8}=2s.\ 6d.$ the value of the purse, and, consequently, $17s.\ 6d.$ the money contained in it.

Ex. 3. Let the number of sheep $=x$; then, by the question, $x+x+\frac{1}{2}x+7\frac{1}{2}=500$; or $2\frac{1}{2}x=492\frac{1}{2}$; mult. by 2, $5x=985$;
Whence, $x=\frac{985}{5}=197$, the number sought.

Ex. 4. Let the length of the post $=x$; then, by the question, $\frac{1}{4}x+\frac{1}{3}x+10=x$; multiply by 12, $3x+4x+120=12x$;
Transposing, $5x=120$, or $x=\frac{120}{5}=24$ feet, the answer required.

Ex. 5. Let the number of guineas $=x$; then, $x-\frac{1}{4}x-\frac{3}{20}x=72$, or multiply by 20, $20x-5x-3x=1440$; whence $12x=1440$, or $x=\frac{1440}{12}=120$ guineas.

Ex. 6. Let B's share $=x$, then, by the question,
A's share $=2x$,
B's share $=3x$,
Consequently, $x+2x+3x=300$, or $6x=300$; whence $x=\frac{300}{6}=50l$. B's share, $2x=100l$. A's share, and $3x=150l$. C's share.

Ex. 7. Let the age of the wife at the time of the marriage $=x$, and that of the husband $3x$;
Then, after 15 years, their ages will be $x+15$, and $3x+15$; and, by the question, $3x+15=2(x+15)$; or $3x+15=2x+30$;
Whence, by transposing, $x=15$, the age of the wife; and $3x=45$, the age of the husband.

Ex. 8. Let the number sought $=x$; then, by the question, $x-5$ will be the remainder, and $\frac{2(x-5)}{3}=40$;
whence, $2x-10=120$, or $2x=130$, and consequently, $x=\frac{130}{2}=65$, the number sought.

Ex. 9. Let the less number of voters $=x$, then the greater number $=x+120$, and, by the question,
$x+x+120=1296$.
Whence, $2x=1296-120=1176$; consequently, $x=588$, for one candidate, and $x+120=708$, for the other.

Ex. 10. Let x represent the age of C, then
$3x$ is the age of B, and
$6x$ the age of A.

Now by the question, $x+3x+6x=10x=140$; whence $x=\frac{140}{10}=14$ C's age, $3x=42$, B's age, and $6x=84=$A's age.

Ex. 11. Let the equal sum laid out by each be x: then A leaves off with $x+126$, and B with $x-87$; and by the question $x+126=2(x-87)$; or $x+126=2x-174$,

Whence $x=300l.$ the first stock of each.

Ex. 12. Let the price of the harness $=x$; then the price of the horse $=2x$, and the price of the chaise $=6x$; and by the question $x+2x+6x=60l.$ or $9x=60$;

Whence $x=\frac{60}{9}=6l.$ 13*s.* 4*d.* the value of the harness; $2x=13l.$ 6*s.* 8*d.* the horse; and $6x=40l.$ the chaise.

Ex. 13. Let x denote the number of beggars; then by the question $3x-8$ was the number of pence he had about him;

Which from the other part of the question may also be denoted by $2x+3$;

Whence $3x-8=2x+3$, or, by transposing, $x=11$, the number of beggars.

Ex. 14. Let x denote the value of the livery; then $x+8$ is the whole amount of his hire for the year, or for 12 months.

Hence, as $12 : 7 :: x+8 : \frac{7x+56}{12}$, the hire for 7 months; but by the question the servant received $x+2\frac{2}{3}$; whence $\frac{7x+56}{12}=x+2\frac{2}{3}$, or $7x+56=12x+32$,

And by transposition $5x=24$, or $x=\frac{24}{5}=4\frac{4}{5}l.=4l.$ 16*s.*

In the preceding examples only one unknown quantity has been employed, but it will be more convenient in many of the following questions to use two or more unknown letters, according to the nature of the question.

Ex. 15. Here let x denote the son's share, and y the daughter's;

Then the value of their shares will obviously have to each other the same ratio as half a crown to a shilling; that is, as 5 to 2.

Hence, then, we have $\left.\begin{array}{r} x : y :: 5 : 2 \\ \text{And } x+y=560 \end{array}\right\}$

From the first $2x=5y$, or $x=\frac{5y}{2}$.

Whence from the second $\frac{5y}{2}+y=560$

or $5y+2y=7y=1120$: whence $y=\frac{1120}{7}=160l.$

the daughter's share, and $x=\frac{5y}{2}=400l.$ the son's share.

Ex. 16. Here it may be observed, that every number consisting of two digits is equal to 10 times the digit in the tens place plus that in the units.

If therefore x be put for the former, and y for the latter, the number itself will be denoted by $10x+y$; and the number with the digits inverted by $10y+x$.

Hence by question, $\left\{\begin{array}{l} 10x+y=4x+4y \\ 10x+y+18=10y+x. \end{array}\right.$

Whence the second equation gives $9x-9y=-18$,
or $x-y=-2$, or $x=y-2$;

Which substituted in the 1st, gives $10(y-2)+y=4(y-2)+4y$.

And this, by multiplication and transposition, becomes
$10y+y-4y-4y=20-8$, or $3y=12$,

Whence $y=\frac{12}{3}=4$, and $x=y-2=2$;

Therefore the number sought is 24.

Ex. 17. Let x represent the equal income of each; then by the questions, A's yearly expenditure is $\frac{4x}{5}$, and that of B, $\frac{4x}{5}+50$;

In four years, therefore, B spends $\frac{16x}{5}+200$; which exceeds his income in the same time (viz. $4x$) by 100; hence we have the equation

$$\frac{16x}{5}+200=4x+100, \text{ or}$$
$$16x+1000=20x+500$$

Whence $4x=500$, or $x=\frac{500}{4}=125l.$ the income sought.

Ex. 18. Let the number of persons in company be x, and the number of shillings each paid $=y$; then xy will be the whole reckoning.

Now had there been three persons more in company, viz. $(x+3)$, each would have paid $(y-1)$ shillings: whence we have

$$(x+3)(y-1)=xy;$$

And from the other conditions of the question,

$$(x-2)(y+1)=xy.$$

Whence, from actual multiplication, these become

$$xy+3y-x-3=xy$$
$$xy-2y+x-2=xy$$

And consequently by addition we have

$$2xy+y-5=2xy,$$

Or, cancelling the $2xy$ on both sides, $y-5=0$, or $y=5$, the number of shillings each paid.

And by subtracting the second equation from the first,

$$5y-2x-1=0,$$

Whence $2x=5y-1$, or $x=\frac{5y-1}{2}=\frac{24}{2}=12$, the number of persons in company.

Ex. 19. Let x denote the money he had about him; then by borrowing x and spending one shilling, he had left $2x-1$.

Also, at the second tavern, after borrowing $2x-1$; he had $4x-2$; but spending one shilling he had left $4x-3$.

At the third tavern, he borrowed $4x-3$, and then had $8x-6$; and after spending one shilling, he had left $8x-7$.

At the fourth tavern, borrowing $8x-7$, he had $16x-14$; but after spending another shilling he had left $16x-15$; which by the question is equal to nothing.

Whence $16x-15=0$, or $x=\frac{15}{16}=0s.\ 11\frac{1}{4}d.$ the money he had at first.

Ex. 20. Let x and y denote the two parts; then by the question,

$$x+y=75, \text{ and}$$
$$3x-7y=15$$

Mult. the first by 3, $3x+3y=225$
Subtract the 2d, $3x-7y=15$

And we have $10y=210$

Whence $y=\frac{210}{10}=21$ the least number,

And $x=75-y=54$ the greatest number.

Ex. 21. Let x=the whole quantity of the mixture;
then $\frac{1}{2}x+25$ was the quantity of spirits, and
$\frac{1}{3}x-5$ the quantity of water.
Which altogether made the whole x; therefore by addition, $\frac{1}{2}x+\frac{1}{3}x+20=x$; or by mult. by 6,
$3x+2x+120=6x$;
Whence, by transposing, $x=120$, and conseq.
$\frac{1}{2}x+25=35$ gallons of spirits,
$\frac{1}{3}x-5=35$ gallons of water.

Ex. 22. Let x= the number of guineas, and
y= the number of moidores;
Then $21x$= the shillings paid in guineas,
And $27y$= the shillings paid in moidores.

Now, the whole number of pieces used being 100, and the number of shillings paid being 2400, we have

$$\left.\begin{array}{r} x+\ \ y=\ 100 \\ 21x+27y=2400 \end{array}\right\}$$

Multiply the first by 27, and we shall have

$$27x+27y=2700.$$

The second $21x+27y=2400$

Hence, by subtraction, $6x=300$, or $x=50$, the number of guineas; and, consequently, $y=100-x$, or $y=50$, the number of moidores.

Ex. 23. Let x be the number of days; then, $14x$ miles will be travelled by one, and $16x$ miles by the other.

Hence, $30x=197$, or $x=\frac{197}{30}=$6d. $13\frac{3}{5}$h. the time required.

Ex. 24. Let x denote the weight of the body; then, $\frac{1}{2}x+9$ is the weight of the head, and, by the question, $x=\frac{1}{2}x+9+9$, or $\frac{1}{2}x=18$;

Whence, $x=36$, the weight of the body, $\frac{1}{2}x+9=27$, the weight of the head, and 9, the weight of the tail; consequently, $36+27+9=72$*lbs*. the weight of the fish.

Another Solution.

Let $2x=$ the weight of the body;
then, $9+x=$ the weight of the tail.

$$\therefore\ 9+9+x=2x;$$

by transposition, $x=18$;

$\therefore$ the fish weighed $36+27+9=27$*lbs*.

Ex. 25. Let $\frac{x}{2}$, $\frac{x}{3}$, and $\frac{x}{4}$, represent the three parts required, and the three latter conditions of the question will be answered;

For, the first multiplied by 2, the second by 3, and the third by 4, will obviously be all equal to x, and therefore, equal to each other.

Hence, then, it only remains to fulfil the equation.

$$\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=10,$$

Or, multiplying by 12, to clear it of fractions,

$$6x+4x+3x=120,$$

Whence, $13x=120$, or $x=\frac{120}{13}$.

Therefore, $\frac{120}{2\times13}=4\frac{8}{13}$, $\frac{120}{3\times13}=3\frac{1}{13}$, and $\frac{120}{4\times13}=2\frac{4}{13}$ are the parts sought.

Ex. 26. Let $2x$, $3x$, and $4x$, be the three parts; then, it is obvious that $\frac{1}{2}$ the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third are equal to each other;

Wherefore, there only remains the equation

$$2x+3x+4x=36,$$

Whence, $9x=36$, or $x=\frac{36}{9}=4$,

Consequently, $2x=8$, $3x=12$, and $4x=16$, are the parts required.

Ex. 27. Let the value of the first horse be x, and of the second y; then, by the question,

$$\left.\begin{array}{r} x+50=2y \\ y+50=3x \end{array}\right\} \text{ or, by transposition,}$$

$$\left.\begin{array}{r} 2y-\ x=50 \\ -y+3x=50 \end{array}\right\}$$

Multiply the latter by 2, and we shall have

$-2y+6x=100$; to which adding

$2y-\ x=\ 50$, we have

$5x=150$, or $x=30l$ the value of the first horse; and, from the second equation, $y=3x-50=40l.$ the value of the second.

Ex. 28. Let $x=$ A's money, and $y=$ B's. Then, when B gives A 5*s.* the latter will have $x+5$, and the former $y-5$; and if A gives B 5*s.* then A will have $x-5$, and B $y+5$.

Now, by the question, $\left\{\begin{array}{l} y+5=2(x-5) \\ x+5=3(y-5) \end{array}\right.$

Or, multiplying and transposing,

$$\left.\begin{array}{r} 2x-\ y=15 \\ -x+3y=20 \end{array}\right\}$$

Multiply the latter by 2, and we shall have

$-2x+6y=40$, to which adding

$2x-\ \ y=15$, the sum gives

$5y=55$, or $y=11$, B's money,

And $2x=15+y$, or $x=\frac{15+y}{2}=\frac{15+11}{2}=13$, A's money.

Ex. 29. Let x and y be the two numbers; then, by the question, the difference is to the sum as 2 : 3, and the sum to the product as 3 : 5; that is,

$$x-y : x+y :: 2 : 3$$
$$x+y : \ xy \ :: 3 : 5$$

Which, by multiplying extremes and means, gives

$$\left.\begin{aligned}3x-3y&=2x+2y\\5x+5y&=3xy\end{aligned}\right\}$$

From the first, $x=5y$; which, substituted in the second, gives

$$25y+5y=15y^2$$

Hence, dividing by y, and transposing,

$15y=30$; or $y=2$, one number,

And $x=5y=10$, the other.

Ex. 30. Let $x=$ the number of shillings he had at first; then, by the question, he lost $\frac{1}{4}x$, and, therefore, had $\frac{3}{4}x$ left, to which he won 3*s*.

After this, he had $\frac{3}{4}x+3=\frac{3x+12}{4}$, of which last he lost one-third, and had then two-thirds of it remaining; viz. $\frac{6x+24}{12}$; to which adding 2*s*. won, he had $\frac{6x+48}{12}$.

Then losing one-seventh of this, he had six-sevenths of it, viz. $\frac{36x+288}{84}$ left; which, by the question, was 12*s*.

Whence $\frac{36x+288}{84}=12$, or $36x+288=1008$;

And, consequently, $36x=720$, or $x=20s$. the money he had at the beginning.

Ex. 31. Let x be the number of leaps the greyhound takes, and y the length of each; then, by the question,

$3 : 4 : : x : \frac{4}{3}x$, the number the hare takes, and

$3 : 2 : : y : \frac{2}{3}y$, the length of each;

But the hare being 50 of her leaps before the greyhound, he has to pass over $\frac{4}{3}x+50$ leaps of the hare. Now, each of these, viz. x and $\frac{4}{3}x+50$, multiplied by their respective lengths, will obviously be equal; viz.

$$xy=(\tfrac{4}{3}x+50)\tfrac{2}{3}y$$

Or, dividing each side by y, and reducing,

$$3x=\tfrac{8}{3}x+100, \text{ or } 9x-8x=300.$$

Whence, $x=300$, the number of leaps the greyhound takes before he catches the hare.

Another Solution.

Let $3x=$ the number of leaps the greyhound must take;

$\therefore 4x=$ the number the hare takes in the same time,

and $4x+50=$ the whole number she takes;

but $2 : 3 : : 3x : 4x+50$;

$\therefore 9x=8x+100$;

by transposition, $x=100$.

Hence, the greyhound must take 300 leaps.

Ex. 32. Put w, x, y, and z, for the four parts; and let m be that quantity to which each part becomes equal after the operations expressed in the question are performed;

Then we shall have $w+x+y+z=90$.

$$\text{Also } \left.\begin{aligned} w+2&=m \\ x-2&=m \\ 2y&=m \\ \frac{z}{2}&=m \end{aligned}\right\} \quad \text{Or} \quad \left\{\begin{aligned} w&=m-2 \\ x&=m+2 \\ y&=\frac{m}{2} \\ z&=2m \end{aligned}\right.$$

Whence adding together the four last equations

$$w+x+y+z=4m+\frac{m}{2}:$$

or substituting 90 for $w+x+y+z$ from the first $4m+\frac{m}{2}=90$, or $9m=180$; whence $m=20$; consequently, $w=m-2=18$; $x=m+2=22$; $y=\frac{m}{2}=10$; and $z=2m=40$, the parts required.

Ex. 3. Let x, y, and z be the three numbers; then by the question $x+y+z=324$, $x:z::5:7$, and $x+z=2y$, or $x+z-2y=0$, now, by subtracting the last equation from the first, we have $3y=324$, or $y=\frac{324}{3}=108$, and, consequently, by substitution and reduction $x=90$, and $z=126$.

Ex. 34. Let x be the number of days the man would be in dricking it by himself; then $\frac{1}{x}$ will be the quantity he drinks in a day, and $\frac{1}{30}$ is the quantity the woman drinks in a day:

Therefore the quantity both drink together is $\frac{1}{x}+\frac{1}{30}$.

But by the question, $12(\frac{1}{x}+\frac{1}{30})=1$, or multiplying by $30x$, $360+12x=30x$; whence, by transposing, $18x=360$, or $x=20$, the number of days sought.

Ex. 35. Let x be the number of men in the side of the less square, and $x+1$ the number in the side of the greater; then x^2 will be the whole number of men in the former, and $(x+1)^2=x^2+2x+1$ in the latter:

Whence x^2+284, and $x^2+2x+1-25$ will each express the whole number of men; from which we have this equation,

$$x^2+2x-24=x^2+284, \text{ or}$$

$$2x=284+24=308$$

Whence $x=154$; and consequently

$x^2+284=24000$, the whole number of men.

Ex. 36. Let x, y, and z, be the number of days in which A, B, and C, respectively would finish the work; then

A will do $\dfrac{1}{x}$ part of it in one day,

B will do $\dfrac{1}{y}$ part, and C $\dfrac{1}{z}$ part.

Then, by the question we shall have

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{8}$$

$$\frac{1}{x}+\frac{1}{z}=\frac{1}{9}$$

$$\frac{1}{y}+\frac{1}{z}=\frac{1}{10}$$

And consequently by addition

$$\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{1}{8}+\frac{1}{9}+\frac{1}{10}=\frac{121}{360}$$

Or by division,

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{121}{720}$$

From this subtracting each of the three first equations, we have

$$\frac{1}{z}=\frac{31}{720}, \text{ or, } z=\frac{720}{31}=23\frac{7}{31} \text{ days for C}$$

$$\frac{1}{y}=\frac{41}{720}, \text{ or, } y=\frac{720}{41}=17\frac{23}{41}= \text{ days for B}$$

$$\frac{1}{x}=\frac{49}{720}, \text{ or, } x=\frac{720}{49}=14\frac{34}{49}= \text{ days for A}$$

QUADRATIC EQUATIONS.

EXAMPLES FOR PRACTICE.

Ex. 1. Given $x^2-8x+10=19$;
By transp. $x^2-8x=9$
Whence $x=4\pm\sqrt{(16+9)}=4\pm 5$,
Therefore $x=9$, or -1

Ex. 2. Given $x^2-x-40=170$; by transp.

$$x^2-x=210$$

Whence $x=\frac{1}{2}\pm\sqrt{(\frac{1}{4}+210)}=\frac{1}{2}\pm\sqrt{\frac{841}{4}}$,

Therefore $x=\frac{1}{2}\pm\frac{29}{2}=15$, or -14.

Ex. 3. Given $3x^2+2x-9=76$; by transp.

$3x^2+2x=85$, or $x^2+\frac{2}{3}x=\frac{85}{3}$; whence

$$x=\frac{-1}{3}\pm\sqrt{(\frac{1}{9}+\frac{85}{3})}=\frac{-1}{3}\pm\sqrt{\frac{256}{9}}$$

Therefore $x=\frac{-1}{3}\pm\frac{16}{3}=5$, or $-\frac{17}{3}$

Ex. 4. Given $\frac{1}{2}x^2-\frac{1}{3}x+7\frac{3}{8}=8$; by transp.

$$\frac{1}{2}x^2-\frac{1}{3}x=\frac{5}{8}, \text{ or } x^2-\frac{2}{3}x=\frac{10}{8}$$

Whence $x=\frac{1}{3}\pm\sqrt{(\frac{1}{9}+\frac{10}{8})}=\frac{1}{3}\pm\sqrt{\frac{49}{36}}$

Therefore $x=\frac{1}{3}\pm\frac{7}{6}=1\frac{1}{2}$, or $-\frac{5}{6}$

Ex. 5. Given $\frac{1}{2}x-\frac{1}{3}\sqrt{x}=22\frac{1}{6}$

Mult. by 2; $x-\frac{2}{3}\sqrt{x}=44\frac{1}{3}$; whence

$$\sqrt{x}=\frac{1}{3}\pm\sqrt{(\frac{1}{9}+\frac{133}{3})}=\frac{1}{3}\pm\sqrt{\frac{400}{9}}$$

Therefore $\sqrt{x}=(\frac{1}{3}\pm\frac{20}{3})=7$, or $\frac{-19}{3}$,

Conseq. $x=7^2=49$, or $(\frac{-19}{3})^2=\frac{361}{9}$.

Ex. 6. Given $x+\sqrt{(5x+10)}=8$

By transposing $\sqrt{(5x+10)}=8-x$,

By squaring $5x+10=64-16x+x^2$,

Where $x^2-21x=-54$; therefore

$$x=\frac{21}{2}\pm\sqrt{(\frac{441}{4}-54)}=\frac{21}{2}\pm\sqrt{\frac{225}{4}}$$

That is $x=\frac{21}{2}\pm\frac{15}{2}=18$,* or 3, the answer.

Ex. 7. Given $(10+x)^{\frac{1}{2}}-(10+x)^{\frac{1}{4}}=2$.

Here, since the first index is double the second, the equation is a quadratic; therefore by the rule

$$(10+x)^{\frac{1}{4}}=\frac{1}{2}\pm\sqrt{(\tfrac{1}{4}+2)}=\frac{1}{2}\pm\sqrt{\frac{9}{4}}=2,$$

Whence $(10+x)^{\frac{1}{4}}=2$; $(10+x)^{\frac{1}{2}}=4$, $10+x=16$, and $x=6$.

Ex. 8. Given $2x^4-x^2+96=99$; by transp.

$$2x^4-x^2=3, \text{ or } x^4-\frac{1}{2}x^2=\frac{3}{2}$$

Whence $x^2=\frac{1}{4}\pm\sqrt{(\frac{1}{16}+\frac{3}{2})}=\frac{1}{4}\pm\sqrt{\frac{25}{16}}$,

Therefore $x^2=\frac{6}{4}$, and $x=\sqrt{\frac{6}{4}}=\frac{1}{2}\sqrt{6}$.

Ex. 9. Given $x^6+20x^3-10=59$; by trans. $x^6+20x^3=69$; whence $x^3=-10\pm\sqrt{(100+69)}=-10\pm13=3$ or -23; and $x=\sqrt[3]{3}$, or $-\sqrt[3]{23}$.

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* This value of x does not answer the condition of the question; because, from the transposed equation, x must be less than 8. See my *Elementary Treatise on Algebra, Theoretical and Practical.*

Ex. 10. Given $3x^{2n}-2x^n+3=11$; by transposition,

$$3x^{2n}-2x^n=8, \text{ or } x^{2n}-\frac{2}{3}x^n=\frac{8}{3}$$

Whence $x^n=\frac{1}{3}\pm\sqrt{(\frac{1}{9}+\frac{8}{3})}=\frac{1}{3}\pm\sqrt{\frac{25}{9}}$

Therefore $x^n=\frac{6}{3}=2$, and $x=2^{\frac{1}{n}}$

Ex. 11. Given $5\sqrt[4]{x}-3\sqrt{x}=1\frac{1}{3}$, or

$$3\sqrt{x}-5\sqrt[4]{x}=-1\frac{1}{3}; \text{ then}$$

By division $x^{\frac{1}{2}}-\frac{5}{3}x^{\frac{1}{4}}=\frac{-4}{9}$

Where $x^{\frac{1}{4}}=\frac{5}{6}\pm\sqrt{(\frac{25}{36}-\frac{4}{9})}=\frac{5}{6}\pm\sqrt{\frac{9}{36}}$

Whence $x^{\frac{1}{4}}=\frac{5}{6}\pm\frac{3}{6}=\frac{4}{3}$, or $\frac{1}{3}$

Conseq. $x=(\frac{4}{3})^4=3\frac{13}{81}$, or $\frac{1}{81}$

Ex. 12. Given $\frac{2}{3}x\sqrt{(3+2x^2)}=\frac{1}{2}+\frac{2}{3}x^2$

Mult. by $\frac{3}{2}$, and we have $x\sqrt{(3+2x^2)}=\frac{3}{4}+x^2$

Squaring, $3x^2+2x^4=\frac{9}{16}+\frac{6}{4}x^2+x^4$,

Whence $x^4+\frac{3}{2}x^2=\frac{9}{16}$; therefore by the rule

$$x^2=\frac{-3}{4}\pm\sqrt{(\frac{9}{16}+\frac{9}{16})}=\frac{3}{4}\pm\sqrt{\frac{18}{16}},$$

or $x^2=\frac{-3}{4}\pm\frac{3}{4}\sqrt{2}$, or $x=\frac{1}{2}\sqrt{(-3\pm3\sqrt{2})}$

Ex. 13. Given $x\sqrt{(\frac{6}{x}-x)}=\frac{1+x^2}{\sqrt{x}}$; mult. by $\sqrt{x}$,

$x\sqrt{(6-x^2)}=1+x^2$; or by squaring

$6x^2-x^4=1+2x^2+x^4$,

Whence $2x^4-4x^2=-1$, or $x^4-2x^2=-\frac{1}{2}$

Therefore $x^2=1\pm\sqrt{(1-\frac{1}{2})}=1\pm\frac{1}{2}\sqrt{2}$,

Consequently $x=\sqrt{(1\pm\frac{1}{2}\sqrt{2})}$.

Ex. 14. Given $\frac{1}{x}\sqrt{(1-x^3)}=x^2$; multiplying by x,

we have $\sqrt{(1-x^3)}=x^3$, or $1-x^3=x^6$

or $x^6+x^3=1$; whence also we have,

$x^3=-\frac{1}{2}\pm\sqrt{(\frac{1}{4}+1)}=\frac{-1}{2}\pm\frac{1}{2}\sqrt{5}$, or

$x=(-\frac{1}{2}\pm\frac{1}{2}\sqrt{5})^{\frac{1}{3}}$

Ex. 15. Given $x\sqrt{(\frac{a}{x}-1)}=\sqrt{(x^2-b^2)}$. By squaring

$ax-x^2=x^2-b^2$, or $2x^2-ax=b^2$,

That is $x^2-\frac{a}{2}x=\frac{b^2}{2}$, or $x=\frac{a}{4}\pm\sqrt{(\frac{a^2}{16}+\frac{b^2}{2})}=\frac{a}{4}\pm\frac{1}{4}\sqrt{}$

$(8b^2+a^2)$

Ex. 16. Given $\sqrt{(1+x-x^2)}-2(1+x-x^2)=\frac{1}{9}$

Here $(1+x-x^2)-\frac{1}{2}(1+x-x^2)^{\frac{1}{2}}=\frac{-1}{18}$,

Therefore $(1+x-x^2)^{\frac{1}{2}}=\frac{1}{4}\pm\sqrt{(\frac{1}{16}-\frac{1}{18})}=$

$\frac{1}{4}\pm\sqrt{\frac{2}{16.18}}=\frac{1}{4}\pm\sqrt{\frac{1}{144}}=\frac{1}{4}\pm\frac{1}{12}=\frac{1}{3}$, or $\frac{1}{6}$,

Consequently $1+x-x^2=(\frac{1}{3})^2=\frac{1}{9}$, or $(\frac{1}{6})^2=\frac{1}{36}$

From the first, we have $x^2-x=\frac{8}{9}$, which gives

$$x=\frac{1}{2}\pm\sqrt{(\frac{1}{4}+\frac{8}{9})}=\frac{1}{2}\pm\frac{1}{6}\sqrt{41}.$$

From the second, we have $x^2-x=\frac{35}{36}$, which gives

$$x=\frac{1}{2}\pm\sqrt{(\frac{1}{4}+\frac{35}{36})}=\frac{1}{2}\pm\frac{1}{6}\sqrt{44}.$$

Ex. 17. Given $\sqrt{(x-\frac{1}{x})}+\sqrt{(1-\frac{1}{x})}=x$,

Mult. by $\sqrt{(x-\frac{1}{x})}-\sqrt{(1-\frac{1}{x})}$, and we have

$x-1=x\sqrt{(x-\frac{1}{x})}-x\sqrt{(1-\frac{1}{x})}$; divide by x,

$1-\frac{1}{x}=\sqrt{(x-\frac{1}{x})}-\sqrt{(1-\frac{1}{x})}$, and from the first,

$x=\sqrt{(x-\frac{1}{x})}+\sqrt{(1-\frac{1}{x})}$; therefore,

$1+x-\frac{1}{x}=2\sqrt{(x-\frac{1}{x})}$, or, which is the same,

$(x-\frac{1}{x})-2\sqrt{(x-\frac{1}{x})}+1=0$; or, by extracting,

$\sqrt{(x-\frac{1}{x})}-1=0$, or $x-\frac{1}{x}=1$, or $x^2-x=1$,

Consequently, $x=\frac{1}{2}\pm\sqrt{(\frac{1}{4}+1)}=\frac{1}{2}\pm\frac{1}{2}\sqrt{5}$.

Ex. 18. Given $x^{4n}-2x^{3n}+x^n=6$;* then this expression is evidently = to $(x^{2n}-x^n)^2-(x^{2n}-x^n)=6$. Put $y=x^{2n}-x^n$; then, $y^2-y=6$, and $y^2-y+\frac{1}{4}=6+\frac{1}{4}=\frac{25}{4}$, by completing the

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* See Note, page 134, in the book, for the method of resolving this, and the equation in the following example into factors; or my Treatise on Algebra.

square; $y-\frac{1}{2}=\frac{5}{2}=2\frac{1}{2}$, by extracting the root. Whence, $y=3$; therefore, $x^{2n}-x^n=3$; and, completing the square, $x^{2n}-x^n+\frac{1}{4}=3\frac{1}{4}=\frac{13}{4}$, or $x^n-\frac{1}{2}=\frac{1}{2}\sqrt{13}$, by extracting the root; therefore, $x^n=\frac{1}{2}+\frac{1}{2}\sqrt{13}$, and $x=\sqrt[n]{(\frac{1}{2}+\frac{1}{2}\sqrt{13})}$, the Ans.

Ex. 19. Given $x^4-2x^3+x=a$. Here, this equation may be expressed thus, $(x^2-x)^2-(x^2-x)=a$; then, $(x^2-x)^2-(x^2-x)+\frac{1}{4}=a+\frac{1}{4}$, by completing the square; $x^2-x-\frac{1}{2}=\pm\sqrt{(a+\frac{1}{4})}$, by extracting the root; and $x^2-x=\frac{1}{2}\pm\sqrt{(a+\frac{1}{4})}$. Again, $x^2-x+\frac{1}{4}=\frac{3}{4}\pm\sqrt{(a+\frac{1}{4})}$, by completing the square; $x-\frac{1}{2}=\sqrt{[\frac{3}{4}\pm\sqrt{(a+\frac{1}{4})}]}$, by extracting the root; therefore, $x=\frac{1}{2}\pm\sqrt{[\frac{3}{4}\pm\sqrt{(a+\frac{1}{4})}]}$, the Ans.

QUESTIONS PRODUCING QUADRATIC EQUATIONS.

QUESTIONS FOR PRACTICE.

Ex. 1. Let x and y represent the two parts; then we have $x+y=40$, and $x^2+y^2=818$,

From the first, $x=40-y$, or $x^2=1600-80y+y^2$,

By substitution, $1600-80y+2y^2=818$, or

$2y^2-80y=-782$, or $y^2-40y=-391$,

Whence, $y=20\pm\sqrt{(400-391)}=20\pm3=23$, or 17,

But $x=40-y=17$, or 23;

Therefore, 17 and 23 are the parts sought.

Ex. 2. Let x represent the number sought;

Then, by the question, $(10-x)x=21$, or $x^2-10x=-21$,

Whence, $x=5\pm\sqrt{(25-21)}=5\pm2=7$, or 3.

Ex. 3. Let x and y be the two parts; then, by the question, we have

$$\left.\begin{array}{r} x+y=24 \\ 35(x-y)=xy \end{array}\right\}$$

Here, $x=24-y$, which, substituted in the second equation, gives $35(24-2y)=y(24-y)$, or

$$840-70y=24y-y^2, \text{ or } y^2-94y=-840,$$

Whence, $y=47\pm\sqrt{(47^2-840)}=47\pm 37=10$, or 84.*

Consequently, $x=24-y=24-10=14$.

Ex. 4. Let one part $=x$; then, the other will be $20-x$, And, by the question, $20(20-x)=x^2$, or $x^2+20x=400$, Therefore, $x=-10\pm\sqrt{(100+400)}=-10+10\sqrt{5}$,

And the other part $=20-x=30-10\sqrt{5}$.

Ex. 5. Let x and y represent the two parts; then, $x+y=60$, and $xy : x^2+y^2 :: 2 : 5$, or

$$x+y=60, \text{ and } \frac{5xy}{2}=x^2+y^2,$$

Squaring the first, $x^2+2xy+y^2=3600$,

Subtract the second, $x^2-2\frac{1}{2}xy+y^2=0$,

We have $4\frac{1}{2}xy=3600$, or $xy=\frac{7200}{9}=800$,

From this we have $x=\frac{800}{y}$; which, substituted in the first,

gives $\frac{800}{y}+y=60$, or $y^2-60y=-800$,

Whence, $y=30\pm\sqrt{(900-800)}=30\pm 10=40$, or 20,

Consequently, $x=60-40=20$, or $60-20=40$.

In all quadratics of this kind, in which x may be changed for y, and y for x, in the original equations, without altering their form, the two values of one of the quantities may be taken for the values of the two quantities sought.

Ex. 6. Here, in order to avoid radicals, let us assume x^2 and y^2 for the two parts; then, by the question,

$$x^2+y^2=146, \text{ and } x-y=6.$$

Which may now be solved the same as Ex. 1. Another method is as follows:

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* This value of y does not answer the conditions of the question.

By squaring the second, $x^2-2xy+y^2=36$
Subt. it from twice the first, $2x^2 \quad +2y^2=292$
And we have $x^2+2xy+y^2=256$
Hence, by extracting $x+y=16$
But $x-y=6$
Whence, by addition, $2x=22$, or $x=11$, and $x^2=121$,
And, by subtraction, $2y=10$, or $y=5$, and $y^2=25$.

Ex. 7. Let x and y represent the two numbers;
Then, by the question, $\left.\begin{array}{r} x+y=20 \\ xy=36 \end{array}\right\}$
And
Squaring the first, $x^2+2xy+y^2=400$
Subtract 4 times the second, $4xy \quad =144$
And we have $x^2-2xy+y^2=256$
Whence, $x-y=16$; and since also $x+y=20$
by addition we have also $2x=36$, or $x=18$
And by subtraction $2y=4$, or $y=2$

But the more direct solution is as follows:

From the second equation $x=\frac{36}{y}$, which substituted in the first gives $\frac{36}{y}+y=20$, or $y^2-20y=-36$.

Whence $y=10\pm\sqrt{(100-36)}=10\pm 8=18$ or 2,
Therefore 18 and 2 are the parts sought.

Ex. 8. Let x and y be the two numbers, and consequently $\frac{1}{x}$ and $\frac{1}{y}$ their reciprocals. Then by the question $x+y=\frac{4}{3}$ and $\frac{1}{x}+\frac{1}{y}=\frac{16}{5}$; where the 2d becomes $x+y=\frac{16xy}{5}$;

Whence $\frac{16xy}{5}=\frac{4}{3}$, or $xy=\frac{20}{48}=\frac{5}{12}$

Consequently $x=\frac{5}{12y}$; which substituted in the first

gives $\frac{5}{12y}+y=\frac{4}{3}$, or $5+12y^2=16y$, or $y^2-\frac{4}{3}y=\frac{-5}{12}$

Whence $y=\frac{2}{3}\pm\sqrt{(\frac{4}{9}-\frac{5}{12})}=\frac{2}{3}\pm\frac{1}{6}=\frac{1}{2}$, or $\frac{5}{6}$,

Therefore $\frac{1}{2}$ and $\frac{5}{6}$ are the quantities sought.

Ex. 9. Let x and y represent the two numbers.

Then by the question $x-y=15$, and $\frac{xy}{2}=y^3$,

The second equation gives $xy=2y^3$, or $x=2y^2$,

Whence by substitution in the first, we have

$$2y^2-y=15, \text{ or } y^2-\tfrac{1}{2}y=\frac{15}{2}; \text{ and hence}$$

$$y=\tfrac{1}{4}\pm\sqrt{(\frac{1}{16}+\frac{15}{2})}=\tfrac{1}{4}\pm\frac{11}{4}=3.$$

Consequently $x=15+y=15+3=18$.

Here the two values of y are not those of x and y, because y is made to represent the less number, and cannot, therefore, be changed for x without altering the conditions of the question.

Ex. 10. Let x and y be the two numbers; then by the question $x-y=5$, and $x^3-y^3=1685$,

By the 1st $x=5+y$, or $x^3=125+75y+15y^2+y^3$

Consequently $125+75y+15y^2+y^3-y^3=1685$,

That is, dividing by 15, $y^2+5y=104$,

Whence $y=\frac{-5}{2}\pm\sqrt{(\frac{25}{4}+104)}$, or $y=\frac{-5}{2}+\frac{21}{2}=8$;

Therefore $x=5+y=5+8=13$.

Consequently 8 and 13 are the numbers required.

Ex. 11. Let x be the number of pieces, and y the shillings that each piece cost; then by the question

$$xy=675, \text{ and } 48x=675+y.$$

From the 1st, $y=\frac{675}{x}$; whence by substitution we have

$$48x=675+\frac{675}{x}, \text{ or } 48x^2-675x=675,$$

$$\text{Or } x^2-\frac{225}{16}x=\frac{225}{16}; \text{ whence}$$

$$x=\frac{225}{32}\pm\sqrt{(\frac{225^2}{32^4}+\frac{225}{16})}=\frac{225}{32}+\frac{255}{32}=15,$$

$$\text{And } y=\frac{675}{x}=\frac{675}{15}=45, \text{ the shillings each cost.}$$

Ex. 12. Let x and y represent the two numbers; then by the question $x(x+y)=77$, or $x^2+xy=77$,

$$\text{And } y(x-y)=12, \text{ or } xy-y^2=12.$$

By subtraction we have $x^2+y^2=65$; whence $y=\sqrt{(65-x^2)}$; which by substitution in the first, gives

$$x^2+x\sqrt{(65-x^2)}=77, \text{ or } \sqrt{(65-x^2)}=\frac{77-x^2}{x}$$

$$\text{And by squaring, } 65-x^2=\frac{77^2-154x^2+x^4}{x^4} \text{ or}$$

$$2x^4-219x^2=-5929, \text{ or } x^4-\frac{219}{2}x^2=-2964\tfrac{1}{2}$$

$$\text{Whence } x^2=\frac{219}{4}\pm\sqrt{(\frac{219^2}{16}-2964\tfrac{1}{2})}, \text{ or}$$

$$x^2=\frac{219}{4}\pm\frac{23}{4}=\frac{242}{4}=60\tfrac{1}{2}, \text{ or } \frac{196}{4}=49,$$

Consequently $x=\sqrt{60\frac{1}{2}}$, or $\sqrt{49}=7$; and adopting the latter $y=\sqrt{(65-x^2)}=\sqrt{(65-49)}=4$; that is, 7 and 4 are the two numbers required.

Ex. 13. Let x represent the number of sheep, and y the shillings each cost, and consequently $y+2$, what they sold for; then by the question we have

$$\left.\begin{array}{r} xy=1200 \\ \text{And } (x-15)(y+2)=1080 \end{array}\right\};$$ the latter

by multiplying, gives $xy+2x-15y-30=1080$,

Or since $xy=1200$, we have $2x-15y=-90$.

Now $y=\frac{1200}{x}$; and $15y=\frac{18000}{x}$; which substituted

in the latter, gives $2x-\frac{18000}{x}=-90$, or

$2x^2-18000=-90x$, or $x^2+45x=9000$,

Whence $x=\frac{-45}{2}\pm\sqrt{(\frac{45^2}{4}+9000)}=\frac{-45}{2}\pm\frac{195}{2}=75$, the

number of sheep, and $\frac{1200}{75}=16s.$ the price each cost.

Ex. 14. Let x = less number, and xy the greater; then $xy\times x=x^2y^2-x^2$, and $x^2y^2+x^2=x^3y^3-x^3$, by the question; and, by division, $y=y^2-1$, and $y^2+1=xy^3-x$. From the first, $y^2-y=1$, and, by completing the square, $y^2-y+\frac{1}{4}=1+\frac{1}{4}=\frac{5}{4}$; therefore $y-\frac{1}{2}=\frac{1}{2}\sqrt{5}$, by extracting the root, or $y=\frac{1}{2}+\frac{1}{2}\sqrt{5}$. Again, $x=\frac{y^2+1}{y^3-1}=\frac{y+2}{2y}$(because $y^2=y+1$, and $y^3=y^2+y=2y+1$,)$=\frac{1}{2}+\frac{1}{y}=\frac{1}{2}+\frac{2}{1+\sqrt{5}}=\frac{1}{2}\sqrt{5}$, and

$xy=\frac{1}{2}\sqrt{5}\times(\frac{1}{2}+\frac{1}{2}\sqrt{5})=\frac{1}{4}(5+\sqrt{5})$, the Answer.

Ex. 15. Let $(m+n)$ and $(m-n)$ represent the two numbers,

Then by the question $(m+n)\ -(m-n)=2n=8$
And $(m+n)^4-(m-n)^4=14560$
Now $(m+n)^4=m^4+4m^3n+6m^2n^2+4mn^3+n^4$
$(m-n)^4=m^4-4m^3n+6m^2n^2-4mn^3+n^4$
Whence by subtraction $8m^3n+8mn^3=14560$,
Or by division, and substituting $n=4$, we have

$$m^3+16m=455$$

Mult. by m, $m^4+16m^2=455m=65\times7m$
Add $49m^2$ to both sides, and it becomes

$$m^4+65m^2=49m^2+65\times7m$$

Therefore by completing the square,

$$m^4+65m^2+\frac{65^2}{4}=(7m)^2+65(7m)+\frac{65^2}{4}$$

Whence $m^2+\frac{65}{2}=7m+\frac{65}{2}$, or $m^2=7m$, or $m=7$;
Consequently $m+n=7+4=11$ one number,
And $m-n=7-4=\ 3$ the other.

Ex. 16. Let x be the whole number of persons, and y the number of shillings each would have had to pay; then after two were gone, the number was only $(x-2)$, and their reckoning $y+10$. Now by the question

$$xy=175,\text{ and }(x-2)(y+10)=175,$$

From the latter $xy+10x-2y-20=175$, or
Since $xy=175$, we have $10x-2y=20$, or $5x-y=10$
But $y=\frac{175}{x}$; therefora $5x-\frac{175}{x}=10$, or $5x^2-10x=175$,

$$\text{or } x^2-2x=35,$$

Whence $x=1+\sqrt{(1+35)}=7$, the number sought.

Ex. 17. Let x be the number of persons at first, and y the shillings each would have received; then $x+2$ was the number at last, and $y-1$ what each actually received: hence the following equations,

$$xy=144,\text{ and }(x+2)(y-1)=144,$$

From the latter $xy+2y-x-2=144$, or since $xy=144$
$2y-x=2$; from the first $y=\frac{144}{x}$; therefore $\frac{288}{x}-x=2$;

$$\text{or } x^2+2x=288 ;$$

Consequently $x=-1\pm\sqrt{(288+1)}=-1+17=16$ Ans.

Ex. 18. Let $\frac{x^2}{y}$, x, y, and $\frac{y^2}{x}$ represent any four numbers in geometrical progression ; then we have

$$\left.\begin{array}{l}\frac{x^2}{y}+x+y+\frac{y^2}{x}=15=a\\ \frac{x^4}{y^2}+x^2+y^2+\frac{y^4}{x^2}=85=b\end{array}\right\}$$

Make $x+y=s$, and $xy=r$; then will

$$x^2+y^2=s^2-2r$$

$$\frac{x^2}{y}+\frac{y^2}{x}=a-s ;$$

$$\frac{x^4}{y^2}+\frac{y^4}{x^2}=(a-s)^2-2r$$

Here equations first and third are drawn from this consideration, that the sum of the squares of any two quantities is equal to the square of their sum minus the double rectangle of them. Whence by adding the first and third, we have

$$\frac{x^4}{y^2}+x^2+y^2+\frac{y^4}{x^2}=b, \text{ or}$$

$$s^2+(a-s)^2-4r=b,$$

And from the second we have $x^3+y^3=xy(a-s)$

$$\text{or } x^3+y^3=r(a-s).$$

Also since $x^3+y^3=(x+y)^3-3xy(x+y)$ or $=s^3-3rs$

We have $r(a-s)=s^3-3rs$, or $r=\frac{s^3}{2s+a}$,

Which value substituted for r in the above, gives

$$s^2+(a-s)^2-\frac{4s^3}{2s+a}=b, \text{ or}$$

$$2s^2-2as+a^2-\frac{4s^3}{2s+a}=b,$$

Whence $4s^3-4as^2+2a^2s+2as^2-2a^2s+a^3-4s^3=2sb+ab,$

Or by reduction and transposition,

$$2as^2+2bs=a^3-ab, \text{ or}$$

$$s^2+\frac{b}{a}s=\frac{1}{2}a^2-\frac{1}{2}b,$$

Consequently $s=-\frac{b}{2a}\pm\sqrt{(\frac{b^2}{4a^2}+\frac{1}{2}a^2-\frac{1}{2}b)},$

Where, by substituting $a=15$ and $b=85$, we obtain $s=6$;

$$\text{but } r=\frac{s^3}{2s+a}=\frac{216}{27}=8$$

Hence then $x+y=6$, and $xy=8$; from which are determined $x=2$, and $y=4$; and therefore the numbers sought will be 1, 2, 4, and 8.

Ex. 19. Let $m+n$ be one of the numbers, and $m-n$ the other; then we shall have

$$(m+n)+(m-n)=11, \text{ or } 2m=11$$
$$(m+n)^5+(m-n)^5=17831=a.$$

Now $(m+n)^5=m^5+5m^4n+10m^3n^2+10m^2n^3+5mn^4+n^5,$
and $(m-n)^5=m^5-5m^4n+10m^3n^2-10m^2n^3+5mn^4-n^5.$

Consequently by addition we have

$$10mn^4+20m^3n^2+2m^5=a, \text{ or}$$

$$n^4+2m^2n^2=\frac{a-2m^5}{10m}$$

Or substituting for m its value, found as above, in the second term, we have

$$n^4+\frac{121}{2}n^2=\frac{a-2m^5}{10m}$$

$$\text{Whence } n^2=\frac{-121}{4}\pm\sqrt{(\frac{14641}{16}+\frac{a-2m^5}{10m})}$$

Whence, by substituting for a and m, we derive

$$n^2=\tfrac{9}{4}, \text{ or } n=\tfrac{3}{2}; \text{ consequently,}$$
$$m+n=5\tfrac{1}{2}+1\tfrac{1}{2}=7, \text{ and } 5\tfrac{1}{2}-1\tfrac{1}{2}=4,$$

That is, the two numbers are 4 and 7.

Ex. 20. Let $x-3y$, $x-y$, $x+y$, and $x+3y$ be the four numbers; then, by the question,

$$(x-3y)(x-y)(x+y)(x+3y)=176985=a,$$
or $(x^2-9y^2)(x^2-y^2)=a$, or $x^4-10x^2y^2+9y^4=a$;

Or, since by the question $(x+3y)-(x+y)=2y=4$, or $y=2$, this becomes

$$x^4-40x^2=176841;$$

Consequently, $x^2=20\pm\sqrt{(176841+400)}=441$,

Hence, $x=\sqrt{441}=21$, and $x-3y=15$, $x-y=19$, $x+y=23$, and $x+3y=27$, the numbers sought.

Ex. 21. Let $x=$ the number of hours' march of the first detachment, and y the miles per hour; then, $x+1$ will be the hours of the second, and $y-\frac{1}{4}$ the miles per hour.

Whence, by the question, we have

$xy=39$, and $(x+1)(y-\frac{1}{4})=39$, or $xy-\frac{1}{4}x+y-\frac{1}{4}=39$;

Or, since $xy=39$, we have $-\frac{1}{4}x+y-\frac{1}{4}=0$,
or $4y-x=1$.

Again, $x=\dfrac{39}{y}$; whence, $4y-\dfrac{39}{y}=1$, or $4y^2-39=y$,

Or $y^2-\frac{1}{4}y=\frac{39}{4}$; therefore, $y=\frac{1}{8}\pm\sqrt{(\frac{1}{64}+\frac{39}{4})}=\frac{26}{8}=3\frac{1}{4}$.

Consequently, $3\frac{1}{4}$ and 3 miles per hour are their rates of marching.

Ex. 22. Let x and y represent the two numbers; then, by the question,

$$\left.\begin{aligned} x^2+xy&=140 \\ y^2-xy&=78 \end{aligned}\right\}$$

By addition, $x^2+y^2=218$, or $y=\sqrt{(218-x^2)}$.

But, by the first equation, $y=\dfrac{140-x^2}{x}$;

Whence $\frac{140-x^2}{x}=\sqrt{(218-x^2)}$; or by squaring and reducing $19600-280x^2+x^4=218x^2-x^4$, or

$$2x^4-498x^2=-19600,$$

Whence, by dividing by 2, we have

$$x^4-249x^2=-9800;$$

That is, $x^2=\frac{249}{2}\pm\sqrt{(\frac{249^2}{4}-9800)}$, or $x^2=\frac{249}{2}\pm\frac{151}{2}$, viz. $x^2=200$*, or 49; therefore by adopting the latter we shall have $x=7$, and $y=\frac{140-49}{7}=13$; the numbers sought.

Ex. 23. Let x^3= greater, and y^3= less; $x^3-y^3=13\frac{55}{216}$, and $x-y=1\frac{1}{6}$, by the question. Dividing the former equation by the latter $x^2+xy+y^2=\frac{409}{36}$,

And $x^2-2xy+y^2=\frac{49}{36}$, squaring the 2d.

Hence $3xy=\frac{360}{36}=10$, by subtracting.

And, from the second equation $x=1\frac{1}{6}+y$; which value, being substituted for x in the above equation, and we have $3y\times(1\frac{1}{6}+y)=10$, or $y^2+\frac{7}{6}y=\frac{10}{3}$, whence by completing the square, and extracting the root $y=-\frac{7}{12}\pm\sqrt{(\frac{49}{144}+\frac{10}{3})}$, or $y=-\frac{7}{12}+\frac{23}{12}=1\frac{1}{3}$, hence $x=1\frac{1}{6}+1\frac{1}{3}=2\frac{1}{2}$, and, consequently $x^3=15\frac{5}{8}$, and $y^3=2\frac{10}{27}$.

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* This value of x^2 does not answer the condition of the question; because we see from the first equation that x^2 must be less than 140.

Ex. 24. Let $x-y$, x, and $x+y=$ the numbers; then, $(x-y)^2+x^2+(x+y)^2=93$, or $3x^2+2y^2=93$, and $3(x-y)+4x+5(x+y)=66$, or $12x+2y=66$, by the question. From the second equation $y=33-6x$, hence $y^2=1089-396x+36x^2$, which value being substituted for y in the first, we shall have $3x^2+2178-792x+72x^2=93$, or $25x^2-264x=-695$; whence $x=\frac{132}{25}\pm\sqrt{\frac{17424}{625}-\frac{695}{25}}$, or $x=\frac{132}{25}\pm\frac{7}{25}$, hence $x=\frac{139}{25}$, or $x=5$, and by taking the latter value for x, we have $y=33-6x=3$; and, consequently, the numbers are 2, 5, and 8.

Ex. 25. Let x, y, and $z=$ the numbers; then, $x-y:y-z::x:z$; whence $xz-zy=xy-zx$, or $(x+z)y=2xz$. But $x+z=191-y$, and $xz=4032$, by the question;
Therefore $(191-y)y=2\times4032$, or $y^2-191y=-8064$,
Hence $y=63$; by completing the square, &c.

Now $x+z=191-y=191-63=128$, and $xz=4032$.

$$\begin{array}{ll} \text{From the square of the first,} & x^2+2xz+z^2=16384, \\ \text{Take 4 times the second,} & 4xz \quad =16128. \\ \hline \text{And we shall have} & x^2-2xz+z^2= \quad 256. \end{array}$$

Or, $x-z=16$, and $x+z=128$; whence, by addition and subtraction, $x=72$, and $z=56$.

And, consequently, 72, 63, and 56, are the numbers.

Ex. 26. Let x denote the least number, and y the common difference. Then the four numbers will be expressed by x, $x+y$, $x+2y$, and $x+3y$; and the four specified sums by $x+2$, $x+y+4$, $x+2y+8$, and $x+3y+15$.

Whence, by the nature of geometrical proportionals, we have $(x+2)\times(x+2y+8)=(x+y+4)^2$, and $(x+2)\times(x+3y+15)=(x+y+4)\times(x+2y+8)$: that is, $y^2+4y=2x$, and $2y^2+10y+2=5x$, by multiplication and transpos. Hence $5y^2+20y=4y^2+20y+4$; therefore $y^2=4$, or $y=2$; whence $x=6$; and, consequently, the numbers are 6, 8, 10, and 12.

Ex. 27. Let x and $y=$ the numbers; then $(x+y)\times(x-y)=x^2-y^2=5$, and $(x^2+y^2)\times(x^2-y^2)=65$, by the question. By dividing the latter equation by the former,

we have $x^2+y^2=13$; whence, by addition and subtraction, $x=3$, and $y=2$.

Ex. 28. Let $x^2=$ greater, and $y^2=$ less; then, $x^2-x=y^2+y$, and $x^2+y^2=10$, by the question. From the first, by adding $\frac{1}{4}$ to both sides, we have $x^2-x+\frac{1}{4}=y^2+y+\frac{1}{4}$, and, by extracting the square root, $x-\frac{1}{2}=y+\frac{1}{2}$, or $x=y+1$; which value being substituted for x, in the second equation, we have $y^2+2y+1+y^2=10$, or $y^2+y=\frac{9}{2}$, hence, $y=-\frac{1}{2}\pm\frac{1}{2}\sqrt{19}$, and $x=y+1=\frac{1}{2}\pm\sqrt{19}$.

And, consequently, $x^2=5+\frac{1}{2}\sqrt{19}$, and $y^2=5-\frac{1}{2}\sqrt{19}$.

Ex. 29. Let $x=\frac{1}{2}$ sum, and $y=\frac{1}{2}$ difference; then, $x+y$ = greater, and $x-y=$ less. Therefore, $x^2-y^2+2x=61$, and $2x^2+2y^2-2x=88$, by the question; or, dividing the second equation by 2, we have $x^2+y^2-x=44$; hence, by adding this to the first, we have $2x^2+x=105$, or $x^2+\frac{1}{2}x=\frac{105}{2}$.

Whence, $x=-\frac{1}{4}\pm\sqrt{(\frac{1}{16}+\frac{105}{2})}=-\frac{1}{4}\pm\sqrt{\frac{841}{16}}=-\frac{1}{4}\pm\frac{29}{4}=7$; hence, $y=\sqrt{(44-x^2+x)}=\sqrt{(44-49+7)}=\sqrt{2}$.

Therefore, $7+\sqrt{2}$ and $7-\sqrt{2}$ are the numbers.

Ex. 30. Let $x=$ greater, and $y=$ less; then, $(x-y)\times(x^2-y^2)=576$, and $(x+y)\times(x^2+y^2)=2336$, by the question; that is, $x^3-x^2y-xy^2+y^3=576$, and $x^3+x^2y+xy^2+y^3=2336$; hence, $2x^2y+2xy^2=1760$, by subtraction; and, by adding this to the second equation, we have $x^3+3x^2y+3xy^2+y^3=4096$, or $x+y=\sqrt[3]{4096}=16$, by extracting the cube root; but $2x^2y+2xy^2=2xy(x+y)=1760$; then $32xy=1760$, by substitution; or $xy=\frac{1760}{32}=55$.

Whence, the sum and product being given, (by question 7,) we have $x=11$, and $y=5$.

Ex. 31. Let x, y, and $z=$ the numbers; then, $x+y+z=20$, $xz=y^2$, and $x^2+y^2+z^2=140$. From the first equation, $x+z=20-y$, or $x^2+2xz+z^2=400-40y+y^2$; hence,

$x^2+z^2+2y^2=400-40y+y^2$, by substitution, or $x^2+z^2+y^2=400-40y$; therefore $140=400-40y$, $40y=400-140=260$, or $y=6\frac{1}{2}$; and, consequently,

$$x=\frac{20-y+\sqrt{(400-40y-3y^2)}}{2}=6\tfrac{3}{4}+\sqrt{3\tfrac{5}{16}}, \text{ and}$$

$$z=\frac{20-y-\sqrt{(400-40y-3y^2)}}{2}=6\tfrac{3}{4}-\sqrt{3\tfrac{5}{16}}.$$

Ex. 32. Let $x=$ greater, and $y=$ less; then will $xy=320$, and $x^3-y^3 : (x-y)^3 :: 61 : 1$, by the question; hence, dividing by $x-y$, we have $x^2+xy+y^2 : (x-y)^2 :: 61 : 1$; from whence (by subtracting the consequents from their antecedents,) we have $3xy : (x-y)^2 :: 60 : 1$, or $xy : (x-y)^2 :: 20 : 1$; hence, $xy=20(x-y)^2$, or $20(x-y)^2=320$ by substitution, or $(x-y)^2=16$; and, consequently, $x-y=4$. Now $x=y+4$, which value being substituted for x in the first equation, we have $y^2+4y=320$, and, consequently, $y=-2\pm\sqrt{324}=-2\pm18=16$, or -20, and by taking the positive value of y, we have $x=\frac{320}{16}=20$.

NOTE. This can be done *otherwise* by putting $x=\frac{1}{2}$ sum, and $y=\frac{1}{2}$ difference. Then $x+y=$ greater, and $x-y=$ less, from whence the answer can be easily found.

Ex. 33. Let $x=$ the share of A, or the first term of the progression, and let the common ratio $=\frac{1}{y}$, or as 1 to y; then, $x+xy+xy^2+xy^3=700$, and $xy^3-x : xy^2-xy :: m : n$, by the question. From which proportion we have $y^3-1=(y-1)\times\frac{my}{n}$, or $y^2+y+1=\frac{my}{n}$ (by dividing the whole by $y-1$). Hence y is found $=\frac{m-n+\sqrt{(m^2-2mn-3n^2)}}{2n}=\frac{25+7}{24}=\frac{4}{3}$. Whence $x(=\frac{700}{1+y+y^2+y^3}$ from the first equation) is given $=\frac{27\times700}{27+36+48+64}=108$. Therefore the four shares are 108, 144, 192, and 256 dollars.

OF CUBIC EQUATIONS.

To exterminate the second Term from a Cubic Equation.

Ex. 3. Given equation $x^3-6x^2=10$, or $x^3-6x^2-10=0$.

Here, $x=z+2$,

$$\text{Therefore, } \left\{\begin{array}{lr} x^3= & z^3+6z^2+12z+8 \\ -6x^2= & -6z^2-24z-24 \\ -10= & -10 \end{array}\right.$$

Whence, we shall have $z^3-12z-26=0$,
or $z^3-12z=26$, as required.

Ex. 4. Given equation $y^3-15y^2+81y-243=0$.

Here, $y=x+5$,

$$\text{Therefore, } \left\{\begin{array}{lr} y^3= & x^3+15x^2+75x+125 \\ -15y^2= & -15x^2-150x-375 \\ +81y= & +81x+405 \\ -243= & -243 \end{array}\right.$$

Whence, we shall have $x^3+6x-88=0$,
or $x^3+6x=88$, as required.

Ex. 5. Given equation $x^3+\frac{3}{4}x^2+\frac{7}{8}x-\frac{9}{16}=0$.

Here, $x=y-\frac{1}{4}$,

$$\text{Therefore, } \left\{\begin{array}{lr} x^3= & y^3-\frac{3}{4}y^2+\frac{3}{16}y-\frac{1}{64} \\ +\frac{3}{4}x^2= & +\frac{3}{4}y^2-\frac{3}{8}y+\frac{3}{64} \\ +\frac{7}{8}x= & \frac{7}{8}y-\frac{7}{32} \\ -\frac{9}{16}= & \frac{-9}{16} \end{array}\right.$$

Hence, we have $y^3+\frac{11}{16}y-\frac{3}{4}=0$,

or $y^3+\frac{11}{16}y=\frac{3}{4}$, as required.

Ex. 6. Given $x^4+8x^3-5x^2+10x-4=0$.

Here, $x=y-2$,

$$\text{Then, }\left\{\begin{array}{lrrrrr} x^4= & y^4 & -8y^3 & +24y^2 & -32y & +16 \\ 8x^3= & & 8y^3 & -48y^2 & +96y & -64 \\ -5x^2= & & & -5y^2 & +20y & -20 \\ 10x= & & & & +10y & -20 \\ -4= & & & & & -4 \end{array}\right.$$

Whence, we have $y^4-29y^2+94y-92=0$.

Ex. 7. Given $x^4-3x^3+3x^2-5x-2=0$.

Here, $x=y+e$,

$$\text{Then, }\left\{\begin{array}{lrrrrr} x^4= & y^4 & +4ey^3 & +6e^2y^2 & +4e^3y & +e^4 \\ -3x^3= & & -3y^3 & -9ey^2 & -9e^2y & -3e^3 \\ +3x^2= & & & +3y^2 & +6ey & +3e^2 \\ -5x= & & & & -5y & -5e \\ -2= & & & & & -2 \end{array}\right.$$

Now, the value of e, by which the third term is taken away, is had by resolving the quadratic equation $6e^2-9e+3=0$; the roots of which are found $=1$, or $\frac{1}{2}$; hence, by substituting $y+1$ for x, in the given equation, we find $y^4+y^3-4y-6=0$, an equation wanting the third term.

Ex. 8. Given $3x^3-2x+1=0$. Here, let $x=\frac{1}{y}$;

then, $3x^3-2x+1=\frac{3}{y^3}-\frac{2}{y}+1=0$, or $3-2y^2+y^3=0$;

hence, $y^3-2y^2+3=0$; the roots of which are the reciprocals of the former.

Ex. 9. Given $x^4-\frac{1}{2}x^3+\frac{1}{3}x^2-\frac{3}{4}x+\frac{1}{18}=0$. Here let $x=\frac{y}{6}$; then, $\frac{y^4}{6^4}-\frac{y^3}{2.6^3}+\frac{y^2}{3.6^2}-\frac{3y}{4.6}+\frac{1}{18}=y^4-3y^3+12y^2-162y+72=0$, the equation required; the roots of which are six times those of the former.

SOLUTION OF CUBIC EQUATIONS.

Ex. 1. Given $x^3+3x^2-6x=8$, to find x.

Here $x=y-1$,

$$\text{Therefore,}\left\{\begin{array}{rl} x^3= & y^3-3y^2+3y-1 \\ +3x^2= & +3y^2-6y+3 \\ -6x= & -6y+6 \\ -8= & -8 \end{array}\right.$$

Reduced equation $y^3-9y=0$

Consequently $y=0$; and dividing by y, we have $y^2=9$, or $y=+3$, or -3; whence the three values of x are

$$\left.\begin{array}{l} x=y-1=\ \ 0-1=-1 \\ x=y-1=\ \ 3-1=\ \ 2 \\ x=y-1=-3-1=-4 \end{array}\right\}\text{as required.}$$

Ex. 2. Given $x^3+x^2=500$, to find x.

First $x=z-\frac{1}{3}$,

$$\text{Therefore}\left\{\begin{array}{rl} x^3= & z^3-z^2+\frac{1}{3}z-\frac{1}{27} \\ +x^2= & z^2-\frac{2}{3}z+\frac{1}{9} \\ -500= & -500 \end{array}\right.$$

Reduced equation $z^3-\frac{1}{3}z-499\frac{25}{27}=0$, or

$$z^3-\frac{1}{3}z=499\frac{25}{27}$$

Whence we have $a=\frac{-1}{3}$ and $b=499\frac{25}{27}$; consequently by our formula

$$x=\sqrt[3]{\left\{\frac{b}{2}+\sqrt{\left(\frac{b^2}{4}+\frac{a^3}{27}\right)}\right\}}+\sqrt[3]{\left\{\frac{b}{2}-\sqrt{\left(\frac{b^2}{4}+\frac{a^3}{27}\right)}\right\}}$$

becomes

$$=\sqrt[3]{\left\{\frac{499\frac{25}{27}}{2}+\sqrt{\left(\frac{(499\frac{25}{27})^2}{4}-\frac{1}{729}\right)}\right\}}+\sqrt[3]{\left\{\frac{499\frac{25}{27}}{2}-\sqrt{\left(\frac{(499\frac{25}{27})^2}{4}-\frac{1}{729}\right)}\right\}},$$

Which last expression, being reduced, gives

$$z=\sqrt[3]{\left\{249.9629+\sqrt{\left(249.9629^2-\frac{1}{729}\right)}\right\}}+$$
$$\sqrt[3]{\left\{249.9629-\sqrt{\left(249.9629^2-\frac{1}{729}\right)}\right\}}$$

or

$$z=\sqrt[3]{499.9257}+\sqrt[3]{.0001}=7.950,$$

Consequently $x=z-\frac{1}{3}=7.950-.333=7.617$ Answ.

Ex. 3. Given equation $x^3-3x^2=5$.

First $x=y+1$.

$$\text{Therefore}\left\{\begin{array}{rcl} x^3 &=& y^3+3y^2+3y+1 \\ -3x^2 &=& \quad -3y^2-6y-3 \\ -5 &=& \qquad\qquad -5 \end{array}\right.$$

Whence $y^3-3y-7=0$, or $y^3-3y=7$.

Here a being equal to -3, and $b=7$, we shall have, by the formula,

$$y=\sqrt[3]{\left\{\frac{7}{2}+\sqrt{\left(\frac{49}{4}-1\right)}\right\}}+\sqrt[3]{\left\{\frac{7}{2}-\sqrt{\left(\frac{49}{4}-1\right)}\right\}}=\sqrt[3]{6.854102}$$
$$+\sqrt[3]{.145898};$$

or $y=1.899616+.525485=2.425101$:

Hence $x=y+1=2.425101+1=3.425101$.

Ex. 4. The given equation $x^3-6x=6$, being in its proper reduced form, we have

$$x=\sqrt[3]{\{3+\sqrt{(9-8)}\}}+\sqrt[3]{\{3-\sqrt{(9-8)}\}},$$
$$\text{or } x=\sqrt[3]{(3+1)}+\sqrt[3]{(3-1)},$$
$$\text{or } x=\sqrt[3]{4}+\sqrt[3]{2}, \text{ Answer.}$$

Ex. 5. The given equation $x^3+9x=6$ is here already in its reduced form; whence we have immediately

$$x=\sqrt[3]{\{3+\sqrt{(9+27)}\}}+\sqrt[3]{\{3-\sqrt{(9+27)}\}},$$
$$\text{or } x=\sqrt[3]{(3+6)}+\sqrt[3]{(3-6)},$$
$$\text{or } x=\sqrt[3]{9}+\sqrt[3]{-3},$$
$$\text{or } x=\sqrt[3]{9}-\sqrt[3]{3}, \text{ Answer.}$$

Ex. 6. Given $x^3+2x^2-23x=70$. Let $x=y\ \frac{2}{3}$;

$$\text{Therefore}\begin{cases} x^3=y^3-2y^2+\frac{4}{3}y-\frac{8}{27} \\ +2x^2= \quad +2y^2-\frac{8}{3}y+\frac{8}{9} \\ -23x= \quad -23y+\frac{46}{3} \\ -70 \quad -70 \end{cases}$$

Whence $y^3-\frac{73}{3}y-\frac{1460}{27}=0$, assume $y=\frac{z}{3}$; then, $z^3-219z-1460=0$, or $z^3-219z=1460$.

$$z=\sqrt[3]{\{730+\sqrt{[(730)^2-(73)^3]}\}}+\sqrt[3]{\{730-\sqrt{[(730)^2-(73)^3]}\}},$$

or

$z=\sqrt[3]{(730+379.319126)}+\sqrt[3]{(730-379.319126)}$; hence $z=\sqrt[3]{(1109.319126)}+\sqrt[3]{(350.68087)}=10.35188+7.05187$,

and consequently $y=\frac{z}{3}=\frac{17.40375}{3}=5.80125$;

Whence $x=y-\frac{2}{3}=5.80125-.66666=5.13459$ Answer.

Ex. 7. Given equation $x^3-17x^2+54x=350$, to find x.

This equation, by exterminating the second term, becomes

$$z^3-42\frac{1}{3}z=407\frac{15}{27},$$

Whence by the formula, we shall have

$$z=\sqrt[3]{\left\{203\frac{21}{27}+\sqrt{\left[\left(203\frac{21}{27}\right)^2-\left(14\frac{1}{9}\right)^3\right]}\right\}}+$$

$$\sqrt[3]{\left\{203\frac{21}{27}-\sqrt{\left[\left(203\frac{21}{27}\right)^2-\left(14\frac{1}{9}\right)^3\right]}\right\}}$$

Or by reducing and simplifying the expression,

$$z=\frac{1}{3}\sqrt[3]{\left\{5502+\sqrt{[(5502)^2-(127)^3]}\right\}}+$$

$$\frac{1}{3}\sqrt[3]{\left\{5502-\frac{1}{3}\sqrt{[(5502)^2-(127)^3]}\right\}};$$

$$\text{or } z=\frac{1}{3}\sqrt[3]{[5502+\sqrt{28223621}]}+\frac{1}{3}\sqrt[3]{[5502-}$$

$$\sqrt{28223621}];$$

$$\text{or } z=\frac{1}{3}\sqrt[3]{[5502+5312.1776]}+\frac{1}{3}\sqrt[3]{[5502-}$$

$$5312.1776];$$

$$\text{or } z=\frac{1}{3}\sqrt[3]{10814.1776}+\frac{1}{3}\sqrt[3]{189.8224}=9.287391.$$

Consequently $x=z+\frac{17}{3}=14.954068$, Answer.

Ex. 8. Given $x^3-6x=4$. Here x is readily found, by a few trials, to be equal to -2, and therefore

$$x+2)x^3-6x-4(x^2-2x-2$$
$$x^3+2x^2$$
$$\overline{-2x^2-6x}$$
$$-2x^2-4x$$
$$\overline{-2x-4}$$
$$-2x-4$$
$$\overline{}$$

Whence $x^2-2x-2=0$, or $x^2-2x=2$; the roots of which quadratic are $1+\sqrt{3}$, and $1-\sqrt{3}$; and consequently -2, $1+\sqrt{3}$, and $1-\sqrt{3}$, are the three roots of the proposed equation.

Ex. 9. Given $x^3-5x^2+2x=-12$. Here x is readily found by a few trials, to be equal to 3, and therefore

$$x-3)x^3-5x^2+2x+12(x^2-2x-4$$
$$x^3-3x^2$$
$$\overline{-2x^2+2x}$$
$$-2x^2+6x$$
$$\overline{-4x+12}$$
$$-4x+12$$
$$\overline{}$$

Whence $x^2-2x-4=0$, or $x^2-2x=4$; the roots of which quadratic are $1+\sqrt{5}$, and $1-\sqrt{5}$; and consequently 3, $1+\sqrt{5}$, and $1-\sqrt{5}$, are the three roots of the proposed equation.

SOLUTION OF CUBIC EQUATIONS, BY CONVERGING SERIES.

Ex. 1. Given $x^3+9x=30$. Here $a=9$, and $b=30$, whence $\frac{27b^2}{27b^2+4a^3}=\frac{27\times900}{27\times900+4\times27^2}=\frac{100}{112}=\frac{25}{28}$; and

$$\frac{2b}{\sqrt[3]{[2(27b^2+4a^3)]}}=\frac{60}{\sqrt[3]{[2(27\times900+4\times27^2)]}}=\frac{10}{\sqrt[3]{(252)}}$$

$$=\frac{5\sqrt[3]{(216\times294)}}{126}=\frac{\sqrt[3]{(36750)}}{21}.$$

Consequently, formula first, we shall have

1	1.0000000 (A)
$\frac{2.5}{6.9}\times\frac{25}{28}A$	.1653439 (B)
$\frac{8.11}{12.15}\times\frac{25}{28}B$	.0721898 (C)
$\frac{14.17}{18.21}\times\frac{25}{28}C$	.0405828 (D)
$\frac{20.23}{24.27}\times\frac{25}{28}D$	.0257221 (E)
$\frac{26.29}{30.33}\times\frac{25}{28}E$	.0174913 (F)
$\frac{32.35}{36.39}\times\frac{25}{28}F$	.0124581 (G)
$\frac{38.41}{42.45}\times\frac{25}{28}G$	.0091693 (H)
$\frac{44.47}{48.51}\times\frac{25}{28}H$	.0069160 (I)

$\frac{50.53}{54.57}\times\frac{25}{28}\,I$	.0053164 (K)
$\frac{56.59}{60.63}\times\frac{25}{28}\,K$	.0041481 (L)
$\frac{62.65}{66.69}\times\frac{25}{28}\,L$	.0032775 (M)
$\frac{68.71}{72.75}\times\frac{25}{28}\,M$	.0026163 (N)
$\frac{74.77}{78.81}\times\frac{25}{28}\,N$	.0021068 (O)
$\frac{80.83}{84.87}\times\frac{25}{28}\,O$	.0017091 (P)
$\frac{86.89}{90.93}\times\frac{25}{28}\,P$	.0013950 (Q)
$\frac{92.95}{96.99}\times\frac{25}{28}\,Q$	.0011460 (R)
$\frac{98.101}{102.105}\times\frac{25}{28}\,R$	.0009456 (S)
$\frac{104.107}{108.111}\times\frac{25}{28}\,S$	.0007837 (T)
$\frac{110.113}{114.117}\times\frac{25}{28}\,T$	.0006521 (U)
$\frac{116.119}{120.123}\times\frac{25}{28}\,U$	.0005445 (V)
$\frac{122.125}{126.129}\times\frac{25}{28}\,V$	.0004561 (W)
$\frac{128.131}{132.135}\times\frac{25}{28}\,W$	.0003832 (X)
$\frac{134.137}{138.141}\times\frac{25}{28}\,X$	.0003228 (Y)

$\frac{140.143}{144.147}\times\frac{25}{28}Y$	.0002726 (Z)
$\frac{146.149}{150.153}\times\frac{25}{28}Z$	.0002307 (A)
$\frac{152.155}{156.159}\times\frac{25}{28}A$	.0001955 (B)
$\frac{158.161}{162.165}\times\frac{25}{28}B$	.0001662 (C)
$\frac{164.167}{168.171}\times\frac{25}{28}C$	.0001416 (D)
$\frac{170.173}{174.177}\times\frac{25}{28}D$	.0001206 (E)
$\frac{176.179}{180.183}\times\frac{25}{28}E$	.0001031 (F)
$\frac{182.185}{186.189}\times\frac{25}{28}F$	.0000966 (G)
	1.3770034

Log. 1.3770034	0.138934
Log. $\sqrt[3]{(36750)}$	1.521752
Colog. 21	8.677780
No. 2. 18.	0.338466

Therefore $x=2.18$.

Ex. 2. Given $x^3-2x=5$.

Here $a=2$, and $b=5$.

Then $2\sqrt[3]{\frac{b}{2}}=2\sqrt[3]{\frac{5}{2}}=2\sqrt[3]{\frac{20}{8}}=\sqrt[3]{20}$; and $\frac{27b^2-4a^3}{27b^2}=$ $\frac{27\times25-4\times8}{27\times25}=\frac{643}{675}$. Consequently, by formula second,

we shall have $x=+\sqrt[3]{20}(1-\frac{2}{3.6}\times\frac{643}{675}A-\frac{5.8}{9.12}\times\frac{643}{675}B-\frac{11.14}{15.18}\times\frac{643}{675}C-\frac{17.20}{21.24}\times\frac{643}{675}D-\frac{23.26}{27.30}\times\frac{643}{675}E-\frac{29.32}{33.36}\times\frac{643}{675}F-\frac{35.38}{39.42}\times\frac{643}{675}G-\&c.)$

NOTE. This series converges so slowly, that it requires to calculate more than 60 terms in order to approximate to the answer; which is always the case when $4a^3$ is considerably less than $27b^2$; then the numerator of the multiplier is nearly equal to the denominator, as above, the numerator 643 is nearly equal to 675, and, therefore, the series must converge slowly; consequently, this method is not useful except when $4a^3$ is nearly equal to $27b^2$; then the numerator is small in comparison with the denominator, and the answer is found by summing a few terms of the series. Therefore Cardan's Rule, in Case 2, is more expeditious than this method, when $4a^3$ is considerably less than $27b^2$; as in the example $x^3-2x=5$. Here $a=-2$, and $b=5$; then $x=\sqrt[3]{\left\{\frac{b}{2}+\sqrt{(\frac{b^2}{4}-\frac{a^3}{27})}\right\}}+\sqrt[3]{\left\{\frac{b}{2}-\sqrt{(\frac{b^2}{4}-\frac{a^3}{27})}\right\}}=\sqrt{\left\{\frac{5}{2}+\sqrt{(\frac{25}{4}-\frac{8}{27})}\right\}}+\sqrt{\left\{\frac{5}{2}-\sqrt{(\frac{25}{4}-\frac{8}{27})}\right\}}=\sqrt[3]{\left\{\frac{5}{2}+\sqrt{\frac{643}{108}}\right\}}+\sqrt[3]{\left\{\frac{5}{2}-\sqrt{\frac{643}{108}}\right\}}=\sqrt[3]{(2.5+2.44)}+\sqrt[3]{(2.5-2.44)}=\sqrt[3]{(4.94)}+\sqrt[3]{.06}=1.703+.391=2.094$, Answer.

The same method of reasoning is likewise applicable to Case 1.

Ex. 3. Given $x^3-3x=3$. Here $a=3$, and $b=3$; then $2\sqrt[3]{\frac{b}{3}}=2\sqrt[3]{\frac{3}{2}}=\sqrt[3]{12}$; and $\frac{27b^2-4a^3}{27b^2}=\frac{27.9-4.27}{27.9}=\frac{9-4}{9}=\frac{5}{9}$;

Consequently, by formula 2, we shall have,

1	1.0000000 (A)
$-\frac{2}{3.6}\times\frac{5}{9}A$	—.0617283 (B)
$-\frac{5.8}{9.12}\times\frac{5}{9}B$	—.0127012 (C)
$-\frac{11.14}{15.18}\times\frac{5}{9}C$	—.0040246 (D)
$-\frac{17.20}{21.24}\times\frac{5}{9}D$	—.0015083 (E)
$-\frac{23.26}{27.30}\times\frac{5}{9}E$	—.0006186 (F)
$-\frac{29.32}{33.36}\times\frac{5}{9}F$	—.0002684 (G)
$-\frac{35.38}{39.42}\times\frac{5}{9}G$	—.0001210 (H)
$-\frac{41.44}{45.48}\times\frac{5}{9}H$	—.0000561 (I)
$-\frac{47.50}{51.54}\times\frac{5}{9}I$	—.0000265 (K)
$-\frac{53.56}{57.60}\times\frac{5}{9}K$	—.0000127 (L)
$-\frac{59.62}{63.66}\times\frac{5}{9}L$	—.0000062 (M)
$-\frac{65.68}{69.72}\times\frac{6}{9}M$	—.0000030 (N)
$-\frac{71.74}{75.78}\times\frac{5}{9}N$	—.0000014 (O)

L

$-\frac{77.80}{81.84}\times\frac{5}{9}O$ $\qquad$ $-.0000007$ (P)

Sum,	-0.0810770
Com.	$.9189230$
Log. .9189230,	-1.963279
Log. $\sqrt[3]{12}$,	0.359727
No. 2.1038,	0.323006

Therefore, $x=2.1038$, Answer.

Ex. 4. Given $x^3-27x=36$.

Here $a=27$, and $b=36$.

Then, $\frac{27b^2}{4a^3-27b^2}=\frac{27\times36^2}{4.27^3-27.36^2}=\frac{36^2}{4.27^2-36^2}=\frac{324}{405}=\frac{4}{5}$;

and $\frac{-2b}{\sqrt[3]{[2(4a^3-27b^2)]}}=\frac{-12}{\sqrt[3]{(27^2-324)}}=\frac{-4}{\sqrt[3]{15}}=-\frac{\sqrt[3]{(14400)}}{15}$;

Consequently, by formula 4, we shall have

1	$+1.0000000$ (A)
$-\frac{2.5}{6.9}\times\frac{4}{5}A$	$-.1481481$ (B)
$+\frac{8.11}{12.15}\times\frac{4}{5}B$	$+.0579423$ (C)
$-\frac{14.17}{18.21}\times\frac{4}{5}C$	$-.0276712$ (D)
$+\frac{20.23}{24.27}\times\frac{4}{5}D$	$+.0157145$ (E)
$-\frac{26.29}{30.33}\times\frac{4}{5}E$	$-.0095747$ (F)

$+\frac{32.35}{36.39}\times\frac{4}{5}F$	$+.0061103$ (G)
$-\frac{38.41}{42.45}\times\frac{4}{5}G$	$-.0040295$ (H)
$+\frac{44.47}{48.51}\times\frac{4}{5}H$	$+.0027232$ (I)
$-\frac{50.53}{54.57}\times\frac{4}{5}I$	$-.0018756$ (K)
$+\frac{56.59}{60.63}\times\frac{4}{5}K$	$+.0013113$ (L)
$-\frac{62.65}{66.69}\times\frac{4}{5}L$	$-.0009284$ (M)
$+\frac{68.71}{72.75}\times\frac{4}{5}M$	$+.0006639$ (N)
$-\frac{74.77}{78.81}\times\frac{4}{5}N$	$-.0004790$ (O)
$+\frac{80.83}{84.87}\times\frac{4}{5}O$	$+.0003489$ (P)
$-\frac{86.89}{90.93}\times\frac{4}{5}P$	$-.0002553$ (Q)
$+\frac{92.95}{96.99}\times\frac{4}{5}Q$	$+.0001878$ (R)
$-\frac{98.101}{102.105}\times\frac{4}{5}R$	$-.0001368$ (S)
$+\frac{104.107}{108.111}\times\frac{4}{5}S$	$+.0001030$ (T)
$-\frac{110.113}{114.117}\times\frac{4}{5}T$	$-.0000833$ (U)
$+\frac{116.119}{120.123}\times\frac{4}{5}U$	$+.0000623$ (V)

$-\frac{122.125}{126.129}\times\frac{4}{5}\mathrm{V}$ $\qquad\qquad$ $-.0000467$ (W)

Sum,	$+.8919368$
Log. .8919368,	-1.9503308
Log. $\sqrt[3]{14400}$,	1.3861208
Colog. 15,	8.8239087
No. 1.446,	.1603503

Therefore, -1.446 is one of the negative roots, or values of x.

Again, $\sqrt[6]{\left(\frac{4a^3-27b^2}{4}\right)}=\sqrt[6]{\left(\frac{4\times27-27\times36^2}{4}\right)}=\sqrt[6]{10935}$; and $\frac{27b^2}{4a^3-27b^2}=\frac{4}{5}$. Hence, $+\frac{r}{2}\pm\sqrt[6]{(10935)}\times\left\{1+\frac{2}{3.6}\times\frac{4}{5}\mathrm{A}-\frac{5.8}{9.12}\times\frac{4}{5}\mathrm{B}+\frac{11.14}{15.18}\times\frac{4}{5}\mathrm{C}-\text{\&c.}\right\}=5.7657$, or -4.32, the other two roots of the given equation.

Ex. 5. Given $x^3-48x=-200$. First $x=y+16$, and

$$\text{Then}\left\{\begin{array}{l} x^3=y^3+48y^2+\ 768y+\ 4096 \\ -48x^2=\quad -48y^2-1536y-12288 \\ +200=\qquad\qquad\qquad\qquad +\ 200 \end{array}\right.$$

Reduced equation $y^3-768y-7992=0$, or $y^3-768y=7992$.

Here $a=768$, and $b=7992$; whence $2\sqrt[3]{\frac{b}{2}}=2\sqrt[3]{3996}$;

and $\frac{4a^3-27b^2}{27b^2}=\frac{4.768^3-27.7992^2}{27.7992^2}=\frac{86393600}{1724545728}=\ \ldots\ \frac{1349900}{26996027}$.

Consequently, by formula 3, we shall have

1	1.0000000 (A)
$+\frac{2}{3.6}\times\frac{1349900}{26996027}\mathrm{A}$	$+.0055563$ (B)

$-\frac{5.8}{9.12}\times\frac{1349900}{26996027}B$ $\quad -.0001030$ (C)

$+\frac{11.14}{15.18}\times\frac{1349900}{26996027}C$ $\quad +.0000030$ (D)

$+1.0054563$

Log. 1.0054563	.0023632
Log. $2\sqrt[3]{(3996)}$, or $\sqrt[3]{(31968)}$	1.5015718
No. 31. 91,	1.5089350

Therefore $x=31.91+16=47.91$ Answ.

Ex. 6. Given $x^3-22x=24$. Here $a=22$, and $b=24$.

Then $\frac{-2b}{\sqrt[3]{[2(4a^3-27.24^2)]}}=\frac{-48}{\sqrt[3]{[8.10648-54.576]}}=\frac{-12}{\sqrt[3]{845}}$

$=\frac{-12\sqrt[3]{(845)^2}}{845}=-\frac{12\sqrt[3]{(714025)}}{845}$; and $\frac{27b^2}{4a^3-27b^2}$

$=\frac{15552}{42592-15552}=\frac{15552}{27040}=\frac{486}{845}$.

Hence by formula 4, we shall have,

1	1.0000000 (A)
$-\frac{2.5}{6.9}\times\frac{486}{845}A$	$-.1065088$ (B)
$+\frac{8.11}{12.15}\times\frac{486}{845}B$	$+.0299485$ (C)
$-\frac{14.17}{18.21}\times\frac{486}{845}C$	$-.0108452$ (D)
$+\frac{20.23}{24.27}\times\frac{486}{845}D$	$+.0044273$ (E)
$-\frac{26.29}{30.33}\times\frac{486}{845}E$	$-.0019393$ (F)

$+\frac{32.35}{36.39}\times\frac{486}{845}F$	$+.0008897$ (G)
$-\frac{38.41}{42.45}\times\frac{486}{845}G$	$-.0004218$ (H)
$+\frac{44.47}{48.51}\times\frac{486}{845}H$	$+.0002049$ (I)
$-\frac{50.53}{54.57}\times\frac{486}{845}I$	$-.0000960$ (K)
$+\frac{56.59}{60.63}\times\frac{486}{845}K$	$+.0000483$ (L)
$-\frac{62.65}{66.69}\times\frac{486}{845}L$	$-.0000245$ (M)
$+\frac{68.71}{72.75}\times\frac{486}{845}M$	$+.0000126$ (N)
$-\frac{74.77}{78.81}\times\frac{486}{845}N$	$-.0000068$ (O)
$-\frac{80.83}{84.87}\times\frac{486}{845}O$	$+.0000035$ (P)
$-\frac{86.89}{90.93}\times\frac{486}{845}P$	$-.0000018$ (Q)
	$+.9156906$

Log. .9156906	-1.9617485
Log. 12	1.0791812
Log. $\sqrt[3]{(714025)}$	1.9512378
Co-log. 845	17.0731433
No. 1.162	.0653108

Therefore, one of the negative roots or values of x is $-1.162=-r$.

Again, $\sqrt[6]{\left(\frac{4a^3-27b^2}{4}\right)}=\sqrt[6]{\left(\frac{42592-15552}{4}\right)}=\sqrt[6]{6760}$,

and $\frac{27b^2}{4a^3-27b^2}=\frac{486}{845}$. Hence,

1	1.0000000 (A)
$+\frac{2}{3.6}\times\frac{486}{845}A$	+.0639053 (B)
$-\frac{5.8}{9.12}\times\frac{486}{845}B$	—.0136130 (C)
$+\frac{11.14}{15.18}\times\frac{486}{845}C$	+.0044657 (D)
$-\frac{17.20}{21.24}\times\frac{486}{845}D$	—.0017327 (E)
$+\frac{23.26}{27.30}\times\frac{486}{845}E$	+.0007357 (F)
$-\frac{29.32}{33.36}\times\frac{486}{845}F$	—.0003305 (G)
$+\frac{35.38}{39.42}\times\frac{486}{845}G$	+.0001543 (H)
$-\frac{41.44}{45.48}\times\frac{486}{845}H$	—.0000742 (I)
$+\frac{47.50}{51.54}\times\frac{486}{845}I$	+.0000364 (K)
$-\frac{53.56}{57.60}\times\frac{486}{845}K$	—.0000180 (L)
$+\frac{59.62}{63.66}\times\frac{486}{845}L$	+.0000091 (M)
$-\frac{65.68}{69.72}\times\frac{486}{845}M$	—.0000046 (N)
	+1.0535355

Log. 1.0535355	.0226491
Log. $\sqrt[6]{6760}$	.6383244
No. 4.581	.6609735
Therefore, $\frac{r}{2}$	$+.581$
Last number	± 4.581
Result	$+5.162$
Or,	-4.000

And, consequently, $+5.162$, -1.162, and -4 are the three roots; but the root is only required, and this is the affirmative root 5.162.

OF BIQUADRATIC EQUATIONS.

Ex. 1. Here the given equation, viz.

$$x^4-55x^2-30x+504=0,$$

being of the proper form for solution, we have by the second rule,

$$b=-55,\ c=-30, \text{ and } d=504.$$

The cubic, or reduced equation, therefore, viz.

$$z^3-\left(\frac{1}{12}b^2+d\right)z=\frac{1}{108}b^3+\frac{1}{8}c^2-\frac{1}{3}bd$$

becomes

$$z^3-756\tfrac{1}{12}z=7811\tfrac{107}{108},$$

Where the root of this cubic being found $=31.66666$, or $31\frac{2}{3}$, our two quadratics are

$$x^2+\left(\sqrt{2}\left\{31\tfrac{2}{3}+\frac{55}{3}\right\}\right)x=-\left(31\tfrac{2}{3}-\frac{55}{6}\right)+\sqrt{\left\{\left(31\tfrac{2}{3}-\frac{55}{6}\right)^2-504\right\}}$$

$$x^2-\left(\sqrt{2}\left\{31\tfrac{2}{3}+\frac{55}{3}\right\}\right)x=-\left(31\tfrac{2}{3}-\frac{55}{6}\right)-\sqrt{\left\{\left(31\tfrac{2}{3}-\frac{55}{6}\right)^2-504\right\}}$$

or $x^2+10x=-21$; and $x^2-10x=-24$.

The first gives $x=5\pm\sqrt{(25-21)}$, or $x=3$, and 7.

And the second, $x=-5\pm\sqrt{(25-24)}$, or $x=-6$, and -4;

That is, the four roots are 3, 7, -4, and -6.

Ex. 2. Given equation $x^4+2x^3-7x^2-8x=-12$.

First, in order to exterminate the second term, we have $x=z-\frac{1}{2}$; whence,

$$\begin{array}{rr}
x^4= & z^4-2z^3+\frac{3}{2}z^2-\frac{1}{2}z+\frac{1}{16} \\
+2x^3= & +2z^3-3z^2+\frac{3}{2}z-\frac{1}{4} \\
-7x^2= & -7z^2+7z-\frac{7}{4} \\
-8x= & -8z+4 \\
+12= & +12 \\
\hline
\end{array}$$

Reduced equation $z^4-8\frac{1}{2}z^2+14\frac{1}{16}=0$,

$$\text{or } z^4=8\frac{1}{2}z^2=-14\frac{1}{16}$$

Whence, $z^2=4\frac{1}{4}\pm\sqrt{(\frac{289}{16}-\frac{225}{16})}=\frac{17}{4}\pm\frac{8}{4}=$

$$\frac{25}{4}, \text{ or } \frac{9}{4},$$

Consequently, $z=\pm\sqrt{\frac{25}{4}}=+\frac{5}{2}$, or $\frac{-5}{2}$,

$$\text{and } z=\pm\sqrt{\frac{9}{4}}=+\frac{3}{2}, \text{ or } -\frac{3}{2},$$

And therefore $x=z-\frac{1}{2}$ has the four following values, viz.

$$x=\frac{5}{2}-\frac{1}{2}=\frac{4}{2}=2$$

$$x=\frac{-5}{2}-\frac{1}{2}=\frac{-6}{2}=-3$$

$$x=\frac{3}{2}-\frac{1}{2}=\frac{2}{2}=1$$

$$x=\frac{-3}{2}-\frac{1}{2}=\frac{-4}{2}=-2.$$

Ex. 3. Given $x^4-8x^3+14x^2+4x=8$ to find x.

First $x=z+2$

$$\begin{array}{lrrrrr}
\text{Whence } x^4= & z^4 & +8z^3 & +24z^2 & +32z & +16 \\
-8x^3= & & -8z^3 & -48z^2 & -96z & -64 \\
+14x^2= & & & +14z^2 & +56z & +56 \\
+4x= & & & & 4z & +8 \\
-8= & & & & & -8 \\
\hline
\end{array}$$

Reduced equation $z^4-10z^2-4z+8=0$,

From which we obtain the following cubic,

$$y^3-(\frac{100}{12}+8)\,y=\frac{-1000}{108}+\frac{16}{8}+\frac{80}{3},$$

$$y^3-16\tfrac{1}{3}y=19\frac{11}{27}.$$

The root of which cubic is $y=-1.3333$, or $\frac{-4}{3}$

Therefore the two quadratic equations are

$$z^2+\left\{\sqrt{2(\frac{-4}{3}+\frac{10}{3})}\right\}z=-(\frac{-4}{3}-\frac{10}{6})+\sqrt{\left\{(\frac{-4}{3}-\frac{10}{6})^2-8\right\}}$$

$$z^2-\left\{\sqrt{2(\frac{-4}{3}-\frac{10}{3})}\right\}z=-(\frac{-4}{3}-\frac{10}{6})-\sqrt{\left\{(\frac{-4}{3}-\frac{10}{6})^2-8\right\}}$$

Or $z^2+2z=2$, and $z^2-2z=4$;

Whence the four roots, or values of z, are

$$z=-1+\sqrt{3},\text{ and } z=-1-\sqrt{3}$$
$$z=+1+\sqrt{5},\text{ and } z=+1-\sqrt{5};$$

And since $x=z+2$, we have the four following values of x, viz.

$$x=3+\sqrt{5},\ x=3-\sqrt{5},$$
$$x=1+\sqrt{3},\ x=1-\sqrt{3},\text{ as required.}$$

The student will observe here, that as the quantities under the latter two radicals, in the above quadratics, arise from the square of a negative quantity, their roots must be taken negative, and not affirmative.

Ex. 4. Here the given equation, viz.

$$x^4-17x^2-20x-6=0.$$

is already in the proper form for solution, having $b=-17$, $c=-20$, and $d=-6$; whence our cubic is

$$z^3-(\frac{17^2}{12}-6)z=\frac{-17^3}{108}+\frac{20^2}{8}-\frac{17\times6}{3},$$

$$\text{or, } z^3-18\frac{1}{12}z=-29\frac{53}{108},$$

The root of which cubic is $z=2.3333$, or $z=2\frac{1}{3}$.

Whence our quadratics are

$$x^2+\{\sqrt{2(\frac{7}{3}+\frac{17}{3})}\}x=-(\frac{7}{3}-\frac{17}{6})+\sqrt{\{(\frac{7}{3}-\frac{17}{6})^2+6\}}$$

$$x^2-\{\sqrt{2(\frac{7}{3}+\frac{17}{3})}\}x=-(\frac{7}{3}-\frac{17}{6})-\sqrt{\{(\frac{7}{3}-\frac{17}{6})^2+6\}}$$

$$\text{Or } x^2+4x=-2,\text{ and } x^2-4x=3,$$

Therefore the four roots, or values of x, are

$$x=-2+\sqrt{2},\ x=-2-\sqrt{2},$$
$$x=+2+\sqrt{7},\ x=+2-\sqrt{7},\text{ as required.}$$

Ex. 5. Given $x^4-3x^2-4x-3=0$. Here, $b=-3$, $c=-4$, and $d=-3$: whence, by substituting the values in the equation $z^3-(\frac{1}{12}b^2+d)z=\frac{1}{108}b^3+\frac{1}{8}c^2-\frac{1}{3}bd$, and simplifying the results, we shall have $z^3+2\frac{1}{4}z+1\frac{1}{4}=0$; the root of which last equation, as found by trial, or by one of the former rules, is $-\frac{1}{2}=-r$. Whence our quadratics are

$$x^2+\left\{\sqrt{2\left(-\tfrac{1}{2}+1\right)}\right\}x=-\left(-\tfrac{1}{2}-\tfrac{3}{8}\right)+\sqrt{\left\{\left(-\tfrac{1}{2}-\tfrac{1}{2}\right)^2+3\right\}},$$
or $x^2+x=3$;

and $x^2-\left\{\sqrt{2\left(-\tfrac{1}{2}+1\right)}\right\}x=-\left(-\tfrac{1}{2}-\tfrac{3}{8}\right)-\sqrt{\left\{-\tfrac{1}{2}-\tfrac{1}{2}\right.}$
$\left.(^2+3\right\}$, or $x^2-x=-1$:

From the first of which $x=-\frac{1}{2}\pm\frac{1}{2}\sqrt{13}$, and from the second $x=\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$.

Ex. 6. Given $x^4-19x^3+132x^2-302x+200=0$.

Here $a=-19$, $b=132$, $c=-302$ and $d=200$.

Then, by substituting these values, in the cubic equation, (Rule 1,) we shall have, by simplifying the results, $z^3-217\frac{1}{2}z=1362\frac{1}{2}$; whence, by Cardan's rule, $z=8.6112=r$. Hence, by substituting this value of r in the two quadratic equations, (Rule 1,) and reducing the results, we have the following quadratics:

$$\left.\begin{aligned} z^2-5.08724z&=-3.46262 \\ z^2-13.912754z&=-57.75978 \end{aligned}\right\}$$

Whence the four roots of the given equation are determined by solving these two quadratics.

NOTE. The rule given by Hutton, in his Mathematics, for discovering whether the roots of a biquadratic equation are real or imaginary, fails. See Hutt. Math. vol. ii. page 269, first New-York edition. The error in Hutton's rule will appear evident from the above given equation. The same error is likewise committed by *Maclaurin*.

Ex. 7. Given equation $x^4-27x^3+162x^2+356x-1200=0$.

First $x=z+\dfrac{27}{4}$

$$\begin{array}{rrrrr}
x^4= & z^4+27z^3+\dfrac{2187}{8}z^2 & +\dfrac{19683}{16} & z+\dfrac{531441}{256} \\[2ex]
-27x^3= & -27z^3-\dfrac{2187}{4}z & -\dfrac{59049}{16} & z-\dfrac{531441}{64} \\[2ex]
+162x^2= & +162z^2 & +2187 & z+\dfrac{59049}{8} \\[2ex]
+356x= & & +356 & z+2403 \\
-1200= & & & -1200
\end{array}$$

Reduced equation $z^4-111\frac{3}{8}z^2+82\frac{5}{8}z+2356\frac{77}{256}=0$,

From which we draw the following cubic, viz.

$$y^3+1322\frac{154}{256}y=75539\frac{400}{612},$$

And the root of this equation substituted in our quadratics

$$z^2+\{\sqrt{[2(r-\tfrac{1}{3}b)x]}\}=-(r+\tfrac{1}{6}b)+\sqrt{\{(r+\tfrac{1}{6}b)^2-d\}}$$
$$z^2-\{\sqrt{[2(r-\tfrac{1}{3}b)x]}\}=-(r+\tfrac{1}{6}b)-\sqrt{\{(r+\tfrac{1}{6}b)^2-d\}}$$

gives the four following values of z, viz.

$$z=-4.19392\ ;\quad z=-9.25000$$
$$z=\ 6.90306\ ;\quad z=\ 6.54086\ ;$$

But $x=z+6.25$; therefore the values of x are,

$$x=\ 2.05608\ ;\quad x=-\ 3.00000\ ;$$
$$x=13.15306\ ;\quad x=\ 14.79086.$$

Ex. 8. Given $x^4-12x^2+12x-3=0$, to find x.

Here the equation, in its proper form for solution, being $b=-12$, $c=12$, and $d=-3$; our cubic is

$$z^3-(12-3)z=\frac{-1728}{108}+\frac{144}{8}-12,$$

$$\text{or } z^3-9z=-10,$$

Where we have, from inspection, $z=2$; whence the following quadratics :

$$x^2+\{\sqrt{2(2+4)}\}x=-(2-2)+\sqrt{\{(2-2)^2+3\}}$$
$$x^2-\{\sqrt{2(2+4)}\}x=-(2-2)-\sqrt{\{(2-2)^2+3\}}$$

$$\text{Or } x^2+\sqrt{12}.x=\ \sqrt{3}$$
$$x^2-\sqrt{12}.x=-\sqrt{3}$$

$$\text{Whence } x=-\tfrac{1}{2}\sqrt{12}\pm\sqrt{(3+\sqrt{3})}$$
$$x=+\tfrac{1}{2}\sqrt{12}\pm\sqrt{(3-\sqrt{3})}$$

Or the four values of x in numbers are,

$$x=-3.907378\ ;\quad x=.443277\ ;$$
$$x=\ 2.858083\ ;\quad x=.606018.$$

Ex. 9. Given $x^4-36x^2+72x-36=0$. Here $b=-36$, $c=72$, and $d=-36$;

Whence, by substituting these values, in the equation, $z^3-(\frac{1}{12}b^2+d)z=\frac{1}{108}b^3+\frac{1}{8}c^2-\frac{1}{3}bd$, and this reduced becomes $z^3-72z=-216$, where it is evident, by inspection, that $z=6$, and this value substituted for r, in the two quadratics according to Rule 2, we shall have, after simplifying the results, $x^2+6x=6$, and $x^2-6x=-6$: from the first of which $x=-3\pm3.8729836$; and from the second $x=+3\pm 1.7320506$. Therefore, the four roots of the given equation are, .8729836, 1.2679494, 4.7320506, and -6.8729836.

Ex. 10. Given $x^4-12x^3+47x^2-72x+36=0$.

To take away the second term, let $x=y+3$; then the reduced equation becomes $y^4-7y^2-6y=0$. Here $y=0$, or 0 is evidently one of the roots of this last equation; whence it is reduced to $y^3-7y-6=0$: the roots of which as found by a few trials, or by one of the former rules, are -1, -2, and 3: and, consequently, the four roots of the given equation are 1, 2, 3, and 6.

Ex. 11. Given $x^4+24x^3-114x^2-24x+1=0$.

Here, $a=24$, $b=-114$, $c=-24$, and $d=1$; then by substituting these values in the cubic equation, Rule I., and reducing the result, we have $z^3-1228z=-16272$. Let $z=2y$; then $y^3-307y=-2034$. Now the root of this is found by a few trials, or by one of the former rules, to be 9; hence $z=2y=18=r$.

Again, by substituting this value of r in the two quadratics according to Rule I., we shall have, after reducing the results, $x^2+28x=1$, and $x^2-4x=1$; from the first $x=-14\pm\sqrt{197}$, and from the second $x=2\pm\sqrt{5}$.

Ex. 12. Given $x^4-6x^3-58x^2-114x-11=0$. Here $a=-6$, $b=-58$, $c=-114$, and $d=-11$; then by substituting these values in the cubic equation, Rule I, we have, by reducing the results, $z^3=98\frac{1}{3}z=-1671\frac{2}{2}\frac{5}{7}$. Now, the root of this cubic being found, by one of the former rules for cubic equations, and substituted for r in the two quad-

ratic equations, according to Rule I, we shall have the four roots of the equation. Or, the answer, in the form in which it is given in the introduction, is found thus :

Let $k=(\frac{1}{4}ac-d)=182$, $l=\frac{1}{4}c^2+d(\frac{1}{4}a^2-b)=2512$; and then $A^3-\frac{1}{2}bA^2+kA-\frac{1}{2}l=A^3+29A^2+182A-1256=0$: and from this equation, by trial, A will be found $=4$; and then $B=(2A+\frac{1}{4}a^2-b)^{\frac{1}{2}}=\sqrt{75}=5\sqrt{3}$, $C=\dfrac{aA-c}{2B}=3\sqrt{3}$.

And $x=(\pm\frac{1}{2}B-\frac{1}{4}a\pm(\frac{1}{16}a^2\mp\frac{1}{4}aB^2\pm C-A)^{\frac{1}{2}}=\frac{5}{2}\sqrt{3}+\frac{3}{2}\pm\sqrt{(17\pm\frac{21}{2}\sqrt{3})}$ the roots required.

See Simpson's Algebra, page 150.

RESOLUTION OF EQUATIONS.

BY APPROXIMATION.

Ex. 2. Given equation $x^2+20x=100$.

Here a few trials show the root to be nearly 4.1 ; let therefore $x=4.1+z$;

Then $\left.\begin{array}{l} x^2=16.81+8.2z+z^2 \\ 20x=82.\ \ +20z \end{array}\right\}=100$;

Therefore rejecting z^2, we have

$$98.81+28.2z=100;$$

$$\text{or } z=\frac{1.19}{28.2}=.042;$$

And consequently $x=4.1+z=4.1+.042=4.142$;

Whence by repeating the operation, and assuming $x=4.142+z$, four other figures may be obtained, which gives the value of $x=4.1421356$, as required.

Ex. 3. Given $x^3+9x^2+4x=80$.

By trials we find the root nearly $=2$; assuming, therefore, $x=2+z$, we have

$$\left.\begin{array}{l} x^3=\ 8+12z+6z^2+z^3 \\ 9x^2=36+36z+9z^2 \\ 4x\ =\ 8+\ 4z \end{array}\right\}=80;$$

Whence, by rejecting the second and third powers of z, we have

$$52+52z=80\ ;\ \text{or}\ z=\frac{28}{52}=.5$$

And therefore $x=2.5=$ the root, *nearly*.

Assume now $x=2.5+z$, and we have

$$\left.\begin{array}{l} x^3=15.625+18.75z+7.5z^2+z^3 \\ 9x^2=56.25\ +45.z\ \ +9z^2 \\ 4x\ =10.\ \ \ + 4z \end{array}\right\}=80,$$

Whence rejecting the higher powers of z,

$$81.875+67.75z=80$$

$$\text{or } z=-\frac{1.875}{67.75}=-.0276,$$

Where $x=4.5-.0276=2.472$ *nearly*; and if again we were to assume $x=2.472+z$, we should get four other decimals, which would give $x=2.4721359$.

Remark. In the preceding examples we have involved every power of x completely; but it is obvious, that as the higher powers of z are always rejected, it is unnecessary to carry the expansions beyond the second term, as shown in the following solution:

Ex. 4. Given equation $x^4-38x^3+210x^2+538x+289=0$.

Here, since the second term is equal to all the other terms of the equation, it is obvious that x is less than 38; and by a few trials we find it to be nearly 30; let therefore $x=30+z$; then reserving only the first two terms of each expansion, we have

$$\left.\begin{array}{rrr} x^4= & 810000+ & 108000z+\&c. \\ -\ 38x^3= & -1026000- & 102600z-\&c. \\ +210x^2= & 189000+ & 12600z+\&c. \\ +538x= & 16140+ & 538z+\&c. \\ +289= & 289 & \end{array}\right\}=0.$$

Whence $-10571+18538z=0$, or $z=\frac{10571}{18538}=.5$

And $x=30.5$, *nearly*. Assume, therefore, $x=30.5+z$; and we shall have (rejecting the decimals as inconsiderable)

$$\left.\begin{array}{rl} x^4= & 865365+113490z \\ -38x^3= & -1078159-107816z \\ +210x^2= & 195352+\ 12810z \\ +538x\ = & 16409+\quad 538z \\ +289\ = & 289 \end{array}\right\}=0$$

$$\text{Whence, } -744+\ 19022z=0,$$

$$\text{Or } z=\frac{744}{19022}=.039\text{; whence } x=30.53, \textit{ nearly};$$

And, by assuming again $x=30.53+z$, a still nearer approximation may be obtained. But in equations of this kind, in which the co-efficients are very large with regard to the root, the approximations are always very slow, adding no more than one new figure at every operation; whereas, when the co-efficients are small with respect to the root, each operation doubles the number of figures last obtained.

Ex. 5. Given $x^5+6x^4-10x^3-112x^2-207x+110=0$, to find x.

Here, $x=4$, *nearly*; assume, therefore, $x=4+z$; then,

$$\left.\begin{array}{rl} x^5= & 1024+1280z+\&c. \\ +6x^4= & 1536+1536z+\&c. \\ -10x^3= & -\ 640-\ 480z-\&c. \\ -112x^2= & -1792-\ 896z-\&c. \\ -207x\ = & -\ 828-\ 207z-\&c. \\ +110\ = & +\ 110 \end{array}\right\}=0,$$

$$\text{And } -590+1233z=0,\text{ or } z=\frac{590}{1233}=.47,$$

Whence, $x=4.47$, *nearly*. And, by assuming $x=4.47+z$, another approximation may be obtained, till at last we find $x=4.46410161$; but it would be useless to go through the entire operation in this place.

OF APPROXIMATION BY POSITION.

EXAMPLES FOR PRACTICE.

Ex. 1. Given $x^3+10x^2+5x=2600$.

Here, it is soon discovered that x is a little more than 11; let us assume, therefore, $x=11.0$, and $x=11.1$;

Then, by the rule,

$$11.1 = x = 11.0$$

1367.631	$= x^3 =$	1331
1232.1	$=+10x^2 =$	1210
55.5	$=+5x =$	55
2655.231	Results	2596
	Therefore,	
2655.231	11.1	2600
2596.	11.0	2596

Or 59.231 : .1 : : 4 : .00673,

Whence, $x=11.0+.00673=11.00673$, Ans.

Remark. In this example, one of our suppositions approached so near the truth, that we have been enabled to obtain seven places of figures true, in a single operation; which is a degree of approximation very seldom acquired with so little labour.

Ex. 2. Given $2x^4-16x^3+40x^2-30x+1=0$.

Here, x is nearly $=1$; assume, therefore,

$$2 = x = 1$$

Then,	$32=$	$2x^4=$	2
	$-128=$	$-16x^3=$	-16
	$+160=$	$+40x^2=$	$+40$
	$-60=$	$-30x=$	-30
	$+1=$	$+1=$	$+1$
	$+5$	Results	-3

Therefore,

$$\begin{array}{ccc} +5 & 2 & 0 \\ -3 & 1 & -3 \\ \hline \end{array}$$

$$\text{Or } 8 : 1 :: 3 : .3$$

Whence, $x=1.3$, *nearly;* assume, therefore,

$$1.3 = x = 1.2$$

$$\begin{array}{lllll} \text{Then,} & 5.7122 & = 2x^4 & = + & 4.1472 \\ & -35.152 & = -16x^3 & = - & 27.6480 \\ & +67.6 & = +40x^2 & = + & 57.6000 \\ & -39. & = -30x & = - & 36.0000 \\ & +1. & = +1 & = + & 1.0000 \\ \hline & +.1602 & \text{Results} & - & .9008 \end{array}$$

Therefore,

$$\begin{array}{ccc} +.1602 & 1.3 & 0 \\ -.9008 & 1.2 & -.9008 \\ \hline \end{array}$$

$$\text{Or } 1.0610 : .1 :: .9008 : .085$$

Hence, $x=1.2+.085=1.285$, *nearly;* and, by repeating the operation, we have the still nearer approximation 1.284724.

Ex. 3. Given $x^5+2x^4+3x^3+4x^2+5x=54321$.

Here, we find the value of x to be between 8 and 9; assume, therefore,

$$8 = x = 9$$

$$\begin{array}{lrcccr} \text{Then,} & 32768 & = & x^5 & = & 59049 \\ & 8192 & = & 2x^4 & = & 13122 \\ & 1536 & = & 3x^3 & = & 2187 \\ & 256 & = & 4x^2 & = & 324 \\ & 40 & = & 5x & = & 45 \\ \hline & 42792 & & \text{Results} & & 74727 \end{array}$$

Therefore,

$$\begin{array}{ccc} 74727 & 9 & 54321 \\ 42792 & 8 & 42792 \\ \hline \end{array}$$

$$\text{Or } 31935 : 1 :: 11529 : .3$$

Whence, $x=8.3$, *nearly*. And, by assuming $x=8.3$ and 8.4, another approximation will be obtained; but the successive corrections are very small, and will occupy more room than can be devoted to a single example.

Ex. 4. Given $\sqrt[3]{(7x^3+4x^2)}+\sqrt{(20x^2-10x)}=28$.
Assuming here $x=4$ and $x=5$, we have

$$4 = x = 5$$

$$\begin{array}{rcl} 8 & =\sqrt[3]{(7x^3+4x^2)}= & 9.91596 \\ 16.7332 & =\sqrt{(20x^2-10x)}= & 21.21320 \\ \hline 24.7332 & \text{Results} & 31.12916 \end{array}$$

Therefore

$$\begin{array}{ccc} 31.12916 & 5 & 28 \\ 24.73320 & 4 & 24.7332 \\ \hline \end{array}$$

Or $6.39596 : 1 :: 3.2668 : .51$

Whence, we shall have $x=4.51$, *nearly*.

And, by repeating the operation, $x=4.510661$.

Ex. 5. Given $\sqrt{\{144x^2-(x^2+20)^2\}}+\sqrt{\{196x^2-(x^2+24)^2\}}=114$.

Assuming $x=7$, and $x=8$, we have

$$7 = x = 8$$

$$\begin{array}{rcl} 47.9061 & =\sqrt{\{144x^2-(x^2+20)^2\}}= & 46.4758 \\ 65.3834 & =\sqrt{\{196x^2-(x^2+24)^2\}}= & 69.2820 \\ \hline 113.2895 & \text{Results} & 115.7578 \end{array}$$

Therefore

$$\begin{array}{ccc} 115.7578 & 8 & 114 \\ 113.2895 & 7 & 113.2895 \\ \hline \end{array}$$

Or $2.4683 : 1 :: .7105 : .2$,

Whence, $x=7.2$, *nearly*.

Assuming, now, $x=7.2$, which is found too great: let us therefore take $x=7.1$ and 7.2, and we have

$$7.1 = x = 7.2$$

$$47.9791 = \sqrt{\{144x^2 - (x^2+20)^2\}} = 47.9999$$

$$65.9014 = \sqrt{\{196x^2 - (x^2+24)^2\}} = 66.4003$$

113.8805	Results	114.4002

	114.4002	7.2		114		
	113.8805	7.1		113.8805		
Or	.5197 :	.1	: :	.1195	:	.023

Whence, $x = 7.1 + .023 = 7.123$, *nearly.*

And if, for a new operation, there be taken $x = 7.123$ and 7.124, we shall have $x = 7.123883$.

And a further assumption of this kind will about double the number of decimals.

EXPONENTIAL EQUATIONS.

Ex. 2. Given $x^x = 200$, to find x.

Here, we soon find that x is nearly $=5$, let us, therefore, assume $x = 4.8$, and $x = 4.9$.

Then, the log. $2000 = 3.3010300$; and

Log.	$4.8 = 0.6812412$	Log. $4.9 = 0.6901961$
Mult. by	4.8	4.9
	54499296	62117649
	27249648	27607844
	3.26995776	3.38196089

Therefore,

	3.38196	4.9		3.30103		
	3.26995	4.8		3.26995		
Or	.11201 :	.1	: :	.03108	:	.027

Whence, $x = 4.8 + .027 = 4.827$, *nearly.*

And repeating the operation, by assuming $x = 4.827$, and $x = 4.828$, we shall obtain $x = 4.82782263$.

Ex. 3. Here, $(6x)^x=96$; and our formula must therefore be $x\times(\log. 6+\log. x)=\log. 96$.

Where $\log. 96=1.9822712$,

And x is obviously nearly $=2$; assume, therefore, $x=1.8$, and $x=1.9$; then,

Log. 6. =0.77815		Log. 6 =0.77815
Log. 1.8=0.25527		Log. 1.9=0.27875
1.03342		1.05690
1.8		1.9
826736		951210
103342		105690
1.860156	Results	2.008110

Therefore

2.00811	1.9	1.98227	
1.86015	1.8	1.86015	
.14796 :	.1 ::	.12212 :	.0826,

Whence, $x=1.8+.0826=1.8826$, *nearly.*

Which five figures may be doubled by another operation, making $x=1.8826432$.

Ex. 4. Given equation $x^x=123456789$.

Here, after a few trials, or from inspection in a table of powers, we find x is between 8 and 9, but nearer the latter than the former. Assume, therefore, $x=8.6$, and $x=8.7$.

Then, log. 123456789=8.0915149,

Log. 8.6=0.9344935		log. 8.7=0.9395193
Mult. by 8.6		8.7
56069610		65766351
74759480		75161544
8.03664410	Results	8.17381791

Therefore,

8.17381		8.7	8.09151	
8.03664		8.6	8.03664	
.13717	:	.1 ::	.05487 :	.04,

Whence, $x=8.6+.04=8.64$, nearly.

And, repeating the operation, by assuming x equal to 8.64 and 8.641, x is found $=8.6400268$.

Ex. 5. Given $x^x-x=(2x-x^x)^{\frac{1}{x}}$.

This will be more convenient under the form

$$(x^x-x)^x=2x-x^x.$$

Now, in order to find a first approximate assumption, it may be observed that $2x$ must be greater than x^x, or 2 greater than x^{x-1}.

The same will also be obvious if we put the equation under the form $\frac{(x^x-x)^x}{x}+x^{x-1}=2$.

Since, then, x is less than 2, but nearly equal to that number, let us assume $x=1.8$; then, $x^x=2.8806$, and

$$\frac{(x^x-x)^x}{x} = 0.6387$$

$$\text{Also, } x^{x-1} = 1.6004$$

$$\text{1st result } 2.2391$$

And when $x=1.7$, then,

$$x^x=2.4647, \text{ and } \frac{(x_x-x)^x}{x}=0.3728$$

$$\text{Also, } x^{x-1}=1.4498$$

$$\text{2d result } 1.8226$$

Therefore, by the rule,

	2.2391		1.8	2.0000	
	1.8226		1.7	1.8226	
As	.4165	:	.1 ::	.1774 :	.04;

Whence, $x=1.7+.04=1.74$, *nearly*.

And repeating the operation with the assumption $x=1.74$ and $x=1.75$, we find $x=1.747933$.

Also, by another assumption, a still nearer approximate value of x may be determined; and so on, to any degree of accuracy required.

BINOMIAL THEOREM.

Ex. 5. Here, the proposed binomial being $\sqrt{(1+1)}$, or $(1+1)^{\frac{1}{2}}$, we have $P=1$, $Q=\frac{1}{1}=1$, $\frac{m}{n}=\frac{1}{2}$

Whence,

$$P^{\frac{m}{n}}=1^{\frac{m}{n}}=1^{\frac{1}{2}}=1=A,$$

$$\frac{m}{n}AQ=\frac{1}{2}\times\frac{1}{1}\times\frac{1}{1}=\frac{1}{2}=B,$$

$$\frac{m-n}{2n}BQ=\frac{1-2}{4}\times\frac{1}{2}\times\frac{1}{1}=\frac{-1}{2.4}=C,$$

$$\frac{m-2n}{3n}CQ=\frac{1-4}{6}\times\frac{-1}{2.4}\times\frac{1}{1}=\frac{1.3}{2.4.6}=D,$$

$$\frac{m-3n}{4n}DQ=\frac{1-6}{8}\times\frac{1.3}{2.4.6}\times\frac{1}{1}=\frac{-1.3.5}{2.4.6.8}=E,$$

Where the law of continuation is sufficiently obvious; and therefore we have $\sqrt{(1+1)}=\sqrt{2}=$

$$1+\frac{1}{2}-\frac{1}{2.4}+\frac{1.3}{2.4.6}-\frac{1.3.5}{2.4.6.8}+\frac{1.3.5.7}{2.4.6.8.10}-, \&c.$$

Ex. 6. Here, we have to convert $(8-1)^{\frac{1}{3}}$, or $(2^3-1)^{\frac{1}{3}}$, into a series. Make $2=a$; then it becomes $(a^3-1)^{\frac{1}{3}}$;

where $P=a^3$, $Q=\frac{-1}{a^3}$, and $\frac{m}{n}=\frac{1}{3}$,

Whence,

$$P^{\frac{m}{n}}=(a^3)^{\frac{m}{n}}=(a^3)^{\frac{1}{3}}=a=A,$$

$$\frac{m}{n}AQ=\frac{1}{3}\times\frac{a}{1}\times\frac{-1}{a^3}=\frac{-1}{3a^2}=B,$$

$$\frac{m-n}{2n}BQ=\frac{1-3}{6}\times\frac{-1}{3a^2}\times\frac{-1}{a^3}=\frac{-1.2}{3.6.a^5}=C,$$

$$\frac{m-2n}{3n}CQ=\frac{1-6}{9}\times\frac{-1.2}{3.6a^5}\times\frac{-1}{a^3}=\frac{-1.2.5}{3.6.9a^8}=D,$$

Therefore, $\sqrt[3]{(a^3-1)}=$

$$a-\frac{1}{3.a^2}-\frac{1.2}{3.6a^5}-\frac{1.2.5}{3.6.9a^8}-\frac{1.2.5\cdot 9}{3.6.9.12a^{11}}-, \&c.$$

Or, substituting 2 and its powers for a, we have

$$\sqrt[3]{(8-1)}=\sqrt[3]{7}=$$

$$2-\frac{1}{3.2^2}-\frac{1}{3.6.2^4}-\frac{1.5}{3.6.9.2^7}-\frac{1.5.9}{3.6.9.12.2^{10}}-, \&c.$$

Ex. 7. Here again $(243-3)^{\frac{1}{5}}=(3^5-3)^{\frac{1}{5}}=(a^5-3)^{\frac{1}{5}}$, by writing $a=3$; also $P=a^5$, $Q=\frac{-3}{a^5}$, and $\frac{m}{n}=\frac{1}{5}$; whence,

$$P^{\frac{m}{n}}=(a^5)^{\frac{m}{n}}=(a^5)^{\frac{1}{5}}=a=A,$$

$$\frac{m}{n}AQ=\frac{1}{5}\times\frac{a}{1}\times\frac{-3}{a^5}=\frac{-3}{5a^4}=B,$$

$$\frac{m-n}{2n}BQ=\frac{1-5}{10}\times\frac{-3}{5a^4}\times\frac{-3}{a^5}=\frac{-4.3^2}{5.10.a^9}=C,$$

$$\frac{m-2n}{3n}CQ=\frac{1-10}{15}\times\frac{-4.3^2}{5.10.a^9}\times\frac{-3}{a^5}=\frac{-4.9.3^3}{5.10.15a^{14}}=D,$$

Whence, by writing 3 for a, and cancelling the like powers of 3, we have $\sqrt[5]{(243-3)}=\sqrt[5]{240}=$

$$3-\frac{1}{5.3^3}-\frac{4}{5.10.3^7}-\frac{4.9}{5.10.15.3^{11}}-\frac{4.9.14}{5.10.15.20.3^{15}}-, \&c.$$

Ex. 8. Here, $P=a$, $Q=\frac{\pm x}{a}$, and $\frac{m}{n}=\frac{1}{2}$; therefore,

$P^{\frac{m}{n}}=a^{\frac{m}{n}}=a^{\frac{1}{2}}=A$,

$\frac{m}{n}AQ=\frac{1}{2}\times\frac{a^{\frac{1}{2}}}{1}\times\frac{\pm x}{a}=\pm\frac{x}{2a}a^{\frac{1}{2}}=B$,

$\frac{m-n}{2n}BQ=\frac{1-2}{4}\times\frac{\pm x}{2a}a^{\frac{1}{2}}\times\frac{\pm x}{a}=\frac{-x^2}{2.4a^2}a^{\frac{1}{2}}=C$,

$\frac{m-2n}{3n}CQ=\frac{1-4}{6}\times\frac{-x^2}{2.4a^2}a^{\frac{1}{2}}\times\frac{\pm x}{a}=\frac{\pm 3x^3}{2.4.6a^3}a^{\frac{1}{2}}=D$.

$\frac{m-3n}{4n}DQ=\frac{1-6}{8}\times\frac{\pm 3x^3}{2.4.6a^3}a^{\frac{1}{2}}\times\frac{\pm x}{a}=\frac{-3.5x^4}{2.4.6.8a^4}a^{\frac{1}{2}}=E$,

Whence, $(x\pm a)^{\frac{1}{2}}=$

$$a^{\frac{1}{2}}\left\{1\pm\frac{x}{2a}-\frac{x^2}{2.4a^2}\pm\frac{3x^3}{2.4.6a^3}-\frac{3.5x^4}{2.4.6.8a^4}\pm, \&c.\right\}$$

Ex. 9. Here, we have $P=a$, $Q=\frac{\pm b}{a}$, and $\frac{m}{n}=\frac{1}{3}$

Whence,

$P^{\frac{m}{n}}=a^{\frac{m}{n}}=a^{\frac{1}{3}}=A$,

$\frac{m}{n}AQ=\frac{1}{3}\times\frac{a^{\frac{1}{3}}}{1}\times\frac{\pm b}{a}=\frac{\pm b}{3a}a^{\frac{1}{3}}=B$,

$\frac{m-n}{2n}BQ=\frac{1-3}{6}\times\frac{\pm b}{3a}a^{\frac{1}{3}}\times\frac{\pm b}{a}=\frac{-2b^2}{3.6a^2}a^{\frac{1}{3}}=C$,

$\frac{m-2n}{3n}CQ=\frac{1-6}{9}\times\frac{-2b^2}{3.6a^2}\times\frac{\pm b}{a}=\frac{\pm 2.5b^3}{3.6.9a^3}a^{\frac{1}{3}}=D$,

From which the law of continuation is obvious; and therefore we have $(a\pm b)^{\frac{1}{3}}=$

$$a^{\frac{1}{3}}\left\{1\pm\frac{b}{3a}-\frac{2b^2}{3.6a^2}\pm\frac{2.5b^3}{3.6.9a^3}-\frac{2.5.7b^4}{3.6.9.12a^4}\pm, \&c.\right\}$$

Ex. 10. Here, $P=a$, $Q=\frac{-b}{a}$, and $\frac{m}{n}=\frac{1}{4}$,

Therefore,

$$P^{\frac{m}{n}}=a^{\frac{m}{n}}=a^{\frac{1}{4}}=A,$$

$$\frac{m}{n}AQ=\frac{1}{4}\times\frac{a^{\frac{1}{4}}}{1}\times\frac{-b}{a}=\frac{-b}{4a}a^{\frac{1}{4}}=B,$$

$$\frac{m-n}{n}BQ=\frac{1-4}{8}\times\frac{-b}{4a}a^{\frac{1}{4}}\times\frac{-b}{a}=\frac{-3b^2}{4.8a^2}a^{\frac{1}{4}}=C,$$

$$\frac{m-2n}{3n}CQ=\frac{1-8}{12}\times\frac{-3b^2}{4.8a^2}a^{\frac{1}{4}}\times\frac{-b}{a}=\frac{-3.7b^3}{4.8.12a^3}a^{\frac{1}{4}}=D,$$

Where again the law of the series is discovered; and we have $(a-b)^{\frac{1}{4}}=$

$$a^{\frac{1}{4}}\left\{1-\frac{b}{4a}-\frac{3b^2}{4.8a^2}-\frac{3.7b^3}{4.8.12a^3}-\frac{3.7.11b^4}{4.8.12.16a^4}-, \&c.\right\}$$

In the last three examples, we have repeated the fractional root of the first term in every line; but it is obvious that this is not necessary, as we may leave it out of every term; only remembering to introduce it at last as a general multiplier of the series.

Or, we might have put our examples under a different form, as in the following instance:

Ex. 11. Here, the proposed quantity may be put under the form $a^{\frac{2}{3}}\times(1+\frac{x}{a})^{\frac{2}{3}}$; and, therefore, omitting for the present the multiplier $a^{\frac{2}{3}}$, we have

$$P=1, \quad Q=\frac{x}{a}, \text{ and } \frac{m}{n}=\frac{2}{3}; \text{ whence,}$$

$$P^{\frac{m}{n}}=1^{\frac{m}{n}}=1^{\frac{2}{3}}=1=A,$$

$$\frac{m}{n}AQ=\frac{2}{3}\times\frac{1}{1}\times\frac{x}{a}=\frac{2x}{3a}=B,$$

$$\frac{m-n}{n}BQ=\frac{2-3}{6}\times\frac{2x}{3a}\times\frac{x}{a}=\frac{-2x^2}{3.6a^2}=C,$$

$$\frac{m-2n}{3n}CQ=\frac{2-6}{9}\times\frac{-2x^2}{3.6a^2}\times\frac{x}{a}=\frac{2.4x^3}{3.6.9a^3}=D,$$

$$\frac{m-3n}{4n}DQ=\frac{2-9}{12}\times\frac{2.4x^3}{3.6.9a^3}\times\frac{x}{a}=\frac{-2.4.7x^4}{3.6.9.12a^4}=E,$$

Whence, $(a+x)^{\frac{2}{3}}$, or $a^{\frac{2}{3}}(1+\frac{x}{a})^{\frac{2}{3}}=$

$$a^{\frac{2}{3}}\times\left\{1+\frac{2x}{3a}-\frac{2x^2}{3.6a^2}+\frac{2.4x^3}{3.6.9a^3}-\frac{2.4.7x^4}{3.6.9.12a^4}+,\ \&c.\right\}$$

Which, by cancelling the multiplier 2, in the numerator and denominator, becomes

$$a^{\frac{2}{3}}\times\left\{1+\frac{2x}{3a}-\frac{x^2}{9a^2}+\frac{4x^3}{9^2a^3}-\frac{4.7x^4}{9^2.12a^4}+,\ \&c.\right\}$$

Ex. 12. Here, $P=1$, $Q=-x$, and $\frac{m}{n}=\frac{2}{5}$

Whence, we shall have

$$P^{\frac{m}{n}}=1^{\frac{m}{n}}=1^{\frac{2}{5}}=1=A,$$

$$\frac{m}{n}AQ=\frac{2}{5}\times\frac{1}{1}\times\frac{-x}{1}=\frac{-2x}{5}=B,$$

$$\frac{m-n}{n}BQ=\frac{2-5}{10}\times\frac{-2x}{5}\times\frac{-x}{1}=\frac{-2.3x^2}{5.10}=C,$$

$$\frac{m-2n}{3n}CQ=\frac{2-10}{15}\times\frac{-2.3x^2}{5.10}\times\frac{-x}{1}=\frac{-2.3.8x^3}{5.10.15}=D,$$

Therefore $(1-x)^{\frac{2}{5}}=$

$$1-\frac{2x}{5}-\frac{2.3x^2}{5.10}-\frac{2.3.8x^3}{5.10.15}-\frac{2.3.8.13x^4}{5.10.15.20}-\&c.$$

Ex. 13. Here $\frac{1}{(a\pm x)^{\frac{1}{2}}}=(a\pm x)^{-\frac{1}{2}}$; therefore

$$P=a,\ Q=\frac{\pm x}{a},\text{ and }\frac{m}{n}=\frac{-1}{2},$$

Whence we have $P^{\frac{m}{n}}=a^{\frac{m}{n}}=a^{-\frac{1}{2}}=A,$

And by omitting this factor in the subsequent operations,

$\frac{m}{n}\text{AQ}=\frac{-1}{2}\times\frac{\pm x}{a}=\frac{\mp x}{2a}=\text{B},$

$\frac{m-n}{2n}\text{BQ}=\frac{-1-2}{4}\times\frac{\mp x}{2a}\times\frac{\pm x}{a}=\frac{+3x^2}{2.4a^2}=\text{C},$

$\frac{m-2n}{3n}\text{CQ}=\frac{-1-4}{6}\times\frac{+3x^2}{2.4a^2}\times\frac{\pm x}{a}=\frac{\mp 3.5x^3}{2.4.6a^3}=\text{D},$

$\frac{m-3n}{4n}\text{DQ}=\frac{-1-6}{8}\times\frac{\mp 3.5x^3}{2.4.6a^3}\times\frac{\pm x}{a}=\frac{+3.5.7x^4}{2.4.6.8a^4}=\text{E},$

Whence, introducing the general factor $a^{-\frac{1}{2}}$, or $\frac{1}{a^{\frac{1}{2}}}$,

we have $\frac{1}{(a\pm x)^{\frac{1}{2}}}=$

$$\frac{1}{a^{\frac{1}{2}}}\left\{1\mp\frac{x}{2a}+\frac{3x^2}{2.4a^2}\mp\frac{3.5x^3}{2.4.6a^3}+\frac{3.5.7x^4}{2.4.6.8a^4}\mp, \text{\&c.}\right\}$$

Ex. 14. Here, $\frac{a}{(a\pm x)^{\frac{1}{3}}}=a(a\pm x)^{\frac{-1}{3}}$;

therefore, $\text{P}=a$, $\text{Q}=\frac{\pm x}{a}$, and $\frac{m}{n}=\frac{-1}{3}$. Whence,

$\text{P}^{\frac{m}{n}}=a^{\frac{m}{n}}=a^{-\frac{1}{3}}=\frac{1}{a^{\frac{1}{3}}}=\text{A}$, (and, omitting this term,)

$\frac{m}{n}\text{AQ}=\frac{-1}{3}\times\frac{\pm x}{a}=\frac{\mp x}{3a}=\text{B},$

$\frac{m-n}{2n}\text{BQ}=\frac{-1-3}{6}\times\frac{\mp x}{3a}\times\frac{\pm x}{a}=\frac{+4x^2}{3.6a^2}=\text{C},$

$\frac{m-2n}{3n}\text{CQ}=\frac{-1-6}{9}\times\frac{4x^2}{3.6a^2}\times\frac{\pm x}{a}=\frac{\mp 4.7x^3}{3.6.9a^3}=\text{D},$

$\frac{m-3n}{4n}\text{DQ}=\frac{-1-9}{12}\times\frac{\mp 4.7x^3}{3.6.9a^3}\times\frac{\pm x}{a}=\frac{+4.7.10x^4}{3.6.9.12a^4}=\text{E},$

Whence, introducing our two general factors $a\times\frac{1}{a^{\frac{1}{3}}}=a^{\frac{2}{3}}$,

we have $a(a\pm x)^{-\frac{1}{3}}=$

$$a^{\frac{1}{3}}\left\{1\mp\frac{x}{3a}+\frac{4x^2}{3.6a^2}\mp\frac{4.7x^3}{3.6.9a^3}+\frac{4.7.10x^4}{3.6.9.12a^4}\mp,\ \text{\&c.}\right\}$$

Ex. 15. Here, $\frac{1}{(1+x)^{\frac{1}{5}}}=(1+x)^{-\frac{1}{5}}$; consequently,

$$P=1,\ Q=\frac{x}{1},\ \text{and}\ \frac{m}{n}=\frac{-1}{5},$$

Whence,

$$P^{\frac{m}{n}}=1^{\frac{m}{n}}=1^{\frac{-1}{5}}=1=A,$$

$$\frac{m}{n}AQ=\frac{-1}{5}\times\frac{1}{1}\times\frac{x}{1}=\frac{-x}{5}=B,$$

$$\frac{m-n}{2n}BQ=\frac{-1-5}{10}\times\frac{-x}{5}\times\frac{x}{1}=\frac{+6x^2}{5.10}=C,$$

$$\frac{m-2n}{3n}CQ=\frac{-1-10}{15}\times\frac{6x^2}{5.10}\times\frac{x}{1}=\frac{-6.11x^3}{5.10.15}=D;$$

$$\frac{m-3n}{4n}DQ=\frac{-1-15}{20}\times\frac{-6.11x^3}{5.10.15}\times\frac{x}{1}=\frac{+6.11.16x^4}{5.10.15.20}=E;$$

Therefore, $\frac{1}{(1+x)^{\frac{1}{5}}}=$

$$1-\frac{x}{5}+\frac{6x^2}{5.10}-\frac{6.11x^3}{5.10.15}+\frac{6.11.16x^4}{5.10.15.20}-,\ \text{\&c.}$$

Ex. 16. Here, $\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}=\frac{(a+x)^{\frac{1}{2}}}{(a-x)^{\frac{1}{2}}}=\frac{(a+x)^{\frac{1}{2}}}{(a-x)^{\frac{1}{2}}}\times$

$\frac{(a+x)^{\frac{1}{2}}}{(a+x)^{\frac{1}{2}}}=\frac{a+x}{(a^2-x^2)^{\frac{1}{2}}}=(a+x)(a^2-x^2)^{-\frac{1}{2}}$. Whence, omitting, till the expansion is effected, the leading factor $a+x$; we have $P=a^2$, $Q=\frac{-x^2}{a^2}$, and $\frac{m}{n}=\frac{-1}{2}$.

And, consequently,

$$p^{\frac{m}{n}}=(a^2)^{\frac{m}{n}}=(a^2)^{\frac{-1}{2}}=a^{-1}=\frac{1}{a}=A,$$

$$\frac{m}{n}AQ=\frac{-1}{2}\times\frac{1}{a}\times\frac{-x^2}{a^2}=\frac{x^2}{2a^3}=B,$$

$$\frac{m-n}{2n}BQ=\frac{-1-2}{4}\times\frac{x^2}{2a^3}\times\frac{-x^2}{a^2}=\frac{3x^4}{2.4a^5}=C,$$

$$\frac{m-2n}{3n}CQ=\frac{-1-4}{6}\times\frac{3x^4}{2.4a^5}\times\frac{-x^2}{a^2}=\frac{3.5x^6}{2.4.6a^7}=D,$$

$$\frac{m-3n}{4n}DQ=\frac{-1-6}{8}\times\frac{3.5a^6}{2.4.6a^7}\times\frac{-x^2}{a^2}=\frac{3.5.7x^8}{2.4.6.8a^9}=E,$$

Or $(a^2-x^2)^{\frac{-1}{2}}=$

$$\frac{1}{a}+\frac{x^2}{2a^3}+\frac{3x^4}{2.4a^5}+\frac{3.5x^6}{2.4.6a^7}+, \text{\&c.}$$

Mult. by $\quad a+x$

$$1+\frac{x^2}{2a^2}+\frac{3x^4}{2.4a^4}+\frac{3.5x^6}{2.4.6a^6}+, \text{\&c.}$$

$$+\frac{x}{a}+\frac{x^3}{2a^3}+\frac{3x^5}{2.4a^5}+\frac{3.5x^7}{2.4.6a^7}+, \text{\&c.}$$

Answer. $1+\frac{x}{a}+\frac{x^2}{2a^2}+\frac{x^3}{2a^3}+\frac{3x^4}{2.4a^4}+\frac{3x^5}{2.4a^5}+$, &c.

Or, the expression in this example may be reduced to a more simple form by taking

$$\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}=\left(1+\frac{2x}{a-x}\right)^{\frac{1}{2}}.$$

But, in order to have the same answer as above, the terms of the result must be divided by $a-x$, and its powers.

INDETERMINATE ANALYSIS.

PROBLEM I.

EXAMPLES FOR PRACTICE.

Ex. 1. Given equation $3x=8y-16$.

Here $x=\frac{8y-16}{3}=2y-5+\frac{2y-3}{3}$.

Let $\frac{2y-1}{3}=p$; then $2y-1=3p$; or $2y=3p+1$;

Whence $y=\frac{3p+1}{2}=p+\frac{p+1}{1}$; which latter will obviously be a whole number, provided p be taken any odd number. Assuming $p=3$, we have $y=\frac{3p+1}{2}=5$, and $x=\frac{8y-16}{3}=8$, which are the least values sought.

Ex. 2. Given equation $14x=5y+7$.

Here it may be observed, that since $14x$ and 7 are both divisible by 7, $5y$ must be so likewise; consequently y must be divisible by 7. Let therefore $y=7z$, and we have $5y=35z$; which substitute for $5y$, gives the equation

$$14x=35z+7, \text{ or } 2x=5z+1;$$

Whence $x=\frac{5z+1}{2}=2z+\frac{z+1}{2}=wh.$

Therefore $\frac{z+1}{2}=p$, or $z=2p-1$; whence,

$$x=\frac{5z+1}{2}=\frac{10p-4}{2}=5p-2; \text{ and } y=7z=14p-7.$$

Hence the general values of x and y are $x=5p-2$, and $y=14p-7$, where p may be assumed $=1$, or any integer whatever; the former value, viz. $p=1$, gives $x=3$, and $y=7$, the least values.

Ex. 3. Given equation $27x=1600-16y$.

Here again it will be observed, that since both terms on the right-hand side are divisible by 16, the left-hand member, viz. $27x$, must be so likewise; which cannot be except x itself be divisible by 16; make therefore $x=16z$, and our equation becomes

$$27.16z=1600-16y; \text{ or}$$
$$27z=100-y:$$

Whence $y=100-27z$; and $x=16z$, where z may be assumed at pleasure, providing $27z$ is less than 100.

If $z=1$, then $x=16$, and $y=73$,
$z=2$, then $x=32$, and $y=46$,
$z=3$, then $x=48$, and $y=19$,
Which are the only three answers.

Ex. 4. Let $7x$ and $11y$ be the two parts required, then we have $7x+11y=100$.

$$\text{Whence } x=\frac{100-11y}{7}=14-2y+\frac{2+3y}{7},$$

Make now $\frac{2+3y}{7}=d$; and we have $3y=7d-2$, or $y=\frac{7d-2}{3}=2d-1+\frac{d+1}{3}$. Again, make $\frac{d+1}{3}=b$, and we obtain $d=3b-1$.

$$\text{Therefore } y=\frac{7d-2}{3}=\frac{21b-7-2}{3}=7b-3,$$

$$\text{And } x=\frac{100-11y}{7}=\frac{100-77b+33}{7}=19-11b;$$

Where it is obvious that b cannot be taken greater than 1; making therefore $b=1$, we have $x=19-11b=8$, and $y=7b-3=4$; therefore $8\times7=56$ is one part, and $4\times11=44$, the other.

Ex. 5. Given equation $9x+13y=2000$,

$$\text{Whence } x=\frac{2000-13y}{9}=222-y+\frac{2-4y}{9}=wh.$$

Make now $\frac{2-4y}{9}=p$, or $4y=2-9p$, or

$$y=\frac{2-9p}{4}=-2p+\frac{2-p}{4}=wh.$$

Here making $\frac{2-p}{4}=q$, we have $p=2-4q$,

Hence $y=\frac{2-9p}{4}=\frac{2-18+36q}{4}=9q-4$,

and $x=\frac{2000-13y}{9}=\frac{2000-117q+52}{9}=228-13q$,

Where q may be assumed at pleasure, providing only that $13q$ be less than 228; it may therefore be an integer from 1 to 17; which latter therefore denotes the number of possible solutions.

If $q=1$, then $y=9q-4=5$, and $x=228-13q=215$
If $q=2$, then $y=14$, and $x=202$
If $q=3$, then $y=23$, and $x=189$
&c. &c. &c.

Ex. 6. Given equation $11x+5y=254$.

Here $y=\frac{254-11x}{5}=51-2x-\frac{1+x}{5}=wh$;

Whence making $\frac{1+x}{5}=p$, we have $x=5p-1$,

And $y=\frac{254-11x}{5}=\frac{254-55p+11}{5}=53-11p$,

Where p must be assumed less than $\frac{53}{11}$; that is = any number from 1 to 4.

If $p=1$, then $x=5p-1=4$, and $y=53-11p=42$
$p=2$, then $x=9$; and $y=31$
$p=3$, then $x=14$; and $y=20$
$p=4$, then $x=19$; and $y=9$.

Ex. 7. Given equation $17x+19y+21z=400$.

In questions of this kind, in which only the number of solutions is sought, the answer is more readily obtained from the following rule:

Let $ax+by=c$, be any proposed indeterminate equation, and find the value of p and q in the equation, $ap-bq=1$; then the number of possible solutions of the equation $ax+by=c$, is equal to the difference between the integral parts of the fractions $\frac{cp}{b} \backsim \frac{cq}{a}$.*

In our proposed equation, by transposing $21z$, we have $17x+19y=400-21z$; and by giving z the several values 1, 2, 3, 4, &c. we obtain the following set of equations; p being $=9$, and $q=8$;

Equations.	*No. of Solutions.*
$17x+19y=379$;	$\frac{9.379}{19}-\frac{8.379}{17}=1$
$17x+19y=358$;	$\frac{9.358}{19}-\frac{8.358}{17}=1$
$17x+19y=337$;	$\frac{9.337}{19}-\frac{8.337}{17}=1$
$17x+19y=316$;	$\frac{9.316}{19}-\frac{8.316}{17}=1$
$17x+19y=295$;	$\frac{9.295}{19}-\frac{8.295}{17}=1$
$17x+19y=274$;	$\frac{9.274}{19}-\frac{8.274}{17}=1$
$17x+19y=253$;	$\frac{9.253}{19}-\frac{8.253}{17}=0$
$17x+19y=232$;	$\frac{9.232}{19}-\frac{8.232}{17}=0$
$17x+19y=211$;	$\frac{9.211}{19}-\frac{8.211}{17}=0$
$17x+19y=190$;	$\frac{†9.190}{19}-\frac{8.190}{17}=0$

* See Barlow's Theory of Numbers.

† When any of the left-hand fractions are exactly equal to an integer, the quotient must be diminished by an unit.

$17x+19y=169$;	$\frac{9.169}{19}-\frac{8.169}{17}=1$
$17x+19y=148$;	$\frac{9.148}{19}-\frac{8.148}{17}=1$
$17x+19y=127$;	$\frac{9.127}{19}-\frac{8.127}{17}=1$
$17x+19y=106$;	$\frac{9.106}{19}-\frac{8.106}{17}=1$
$17x+19y=\ 85$;	$\frac{9.85}{19}-\frac{8.85}{17}=0$
$17x+19y=\ 64$;	$\frac{9.64}{19}-\frac{8.64}{17}=0$
$17x+19y=\ 43$;	$\frac{9.43}{19}-\frac{8.43}{17}=0$
$17x+19y=\ 22$;	$\frac{9.22}{19}-\frac{8.22}{17}=0$

Total number of solutions $=10$ Ans.

Ex. 8. Given the equation $5x+7y+11z=224$.

Here z may have any value, from 1 to 19, which gives the following set of equations; also in the equation $5p-7q=1$; we have $p=3$ and $q=2$. Hence

Equations.	*No. of Solutions.*
$5x+7y=213$;	$\frac{3.213}{7}-\frac{2.213}{5}=6$
$5x+7y=202$;	$\frac{3.202}{7}-\frac{2.202}{5}=6$
$5x+7y=191$;	$\frac{3.191}{7}-\frac{2.191}{5}=5$
$5x+7y=180$;	$\frac{3.180}{7}-\frac{2.180}{5}=5$

$5x+7y=169$;	$\frac{3.169}{7}-\frac{2.169}{5}=5$
$5x+7y=158$;	$\frac{3.158}{7}-\frac{2.158}{5}=4$
$5x+7y=147$;	$\frac{3.147}{7}-\frac{2.147}{5}=4$
$5x+7y=136$;	$\frac{3.136}{7}-\frac{2.136}{5}=4$
$5x+7y=125$;	$\frac{3.125}{7}-\frac{2.125}{5}=3$
$5x+7y=114$;	$\frac{3.114}{7}-\frac{2.114}{5}=3$
$5x+7y=103$;	$\frac{3.103}{7}-\frac{2.103}{5}=3$
$5x+7y=\ 92$;	$\frac{3.92}{7}-\frac{2.92}{5}=2$
$5x+7y=\ 81$;	$\frac{3.81}{7}-\frac{2.81}{5}=2$
$5x+7y=\ 70$;	$\frac{3.70}{7}-\frac{2.70}{5}=2$
$5x+7y=\ 59$;	$\frac{3.59}{7}-\frac{2.59}{5}=2$
$5x+7y=\ 48$;	$\frac{3.48}{7}-\frac{2.48}{5}=1$
$5x+7y=\ 37$;	$\frac{3.37}{7}-\frac{2.37}{5}=1$
$5x+7y=\ 26$;	$\frac{3.26}{7}-\frac{2.26}{5}=1$
$5x+7y=\ 15$;	$\frac{3.15}{7}-\frac{2.15}{5}=0$

Total number of solutions $=59$

Ex. 9. Let x be the number of half guineas, and y the number of half crowns; then the number of sixpences in each of the former being 21, and in each of the latter 5; also the number of sixpences being 800, we have the following equation:

$$21x+5y=800;$$

Here the equation $21p-5b=1$, gives $p=1$, and $b=4$;

Whence by the rule $\frac{1.800}{5}-\frac{4.800}{21}=7$. Answer.

Where, as in the preceding examples, the first quotient is distinguished by unity being exactly integral.

This rule, which is taken from Barlow's Theory of Numbers, is much shorter than that which depends upon an actual determination of the several solutions: the latter is omitted here as presenting no difficulty to the student who has attended to the preceding solutions.

Ex. 10. Let x represent the number of guineas I have to give, and y the number of louis-d'ors I am to receive; then by the question,

$$21x-17y=1.$$

Hence $17y=21x-1$, or $y=x+\frac{4x-1}{17}=wh.$

Make $\frac{4x-1}{17}=p$; then $4x=17p+1$,

$$\text{or } x=4p+\frac{p+1}{4}=wh.$$

Where, if we make $\frac{p+1}{4}=b$; then $p=4b-1$,

And consequently $x=4p+\frac{p+1}{4}=17p-4$,

Whence also we have $y=x+\frac{4x-1}{17}=21p-5$;

Where p may be taken any number at pleasure: if we take $p=1$; then $x=13$, and $y=16$, which are the least numbers, viz. I must give 13 guineas, and receive 16 louis.

Ex. 11. Let x, y, and z, be the number of gallons of each sort respectively; then by the question

$$x+y+z=1000$$
$$12x+15y+18z=17(x+y+z)$$

By transposing the terms of the latter equation, we have

$$-5x-2y+z=0$$

But $$x+y+z=1000$$

By subtraction, $$6x+3y=1000$$

Where it is obvious the answer cannot be obtained in integers, because the first side of the equation is divisible by 3, and the other is not. We may therefore assume x or y at pleasure; taking for x the value given in the answer of the Introduction, viz. $x=111\frac{1}{9}$, we have

$$x=111\frac{1}{9};\ y=\frac{1000-6\times111\frac{1}{9}}{9}=\frac{333\frac{1}{3}}{3}=111\frac{1}{9}$$

$$\text{and } z=5x+2y=777\frac{7}{9};$$

That is, $111\frac{1}{9}$ at 12*s.*; $111\frac{1}{9}$ at 15*s.*; and $777\frac{7}{9}$ at 18*s.*

INDETERMINATE ANALYSIS.

PROBLEM II.

Ex. 3. Let $x=$ the number sought; then by the question

$$\frac{x-2}{6}, \text{ and } \frac{x-3}{13}= \text{whole numbers.}$$

$$\text{Make } \frac{x-2}{6}=p; \text{ then } x=6p+2.$$

Substitute this value of x in the second equation, and we have

$$\frac{6p+2-3}{13}=\frac{6p-1}{13}=wh.$$

Make $\frac{6p-1}{13}=q$; then $p=\frac{13q+1}{6}=2q+\frac{q+1}{6}=wh.$

Let now $\frac{q+1}{6}=r$; then $q=6r-1$; and $p=\frac{13(6r-1)+1}{6}$ $=13r-2$; and consequently $x=6(13r-2)+2=78r-10$; where r may be taken at pleasure.

If $r=1$, then $x=68$.

Ex. 4. Let x be the number sought; then by the question

$$\frac{x-5}{7}, \text{ and } \frac{x-2}{9} = \text{ whole numbers,}$$

Make $\frac{x-5}{7}=p$, or $x=7p+5$.

Substitute this value for x in the second equation, and we have

$$\frac{7p+5-2}{9}=\frac{7p+3}{9}=wh.$$

Make $\frac{7p+3}{9}=q$; and we have $p=\frac{9q-3}{7}=q+$

$$\frac{2q-3}{7}=wh.$$

Let now $\frac{2q-3}{7}=r$, and we shall have $q=\frac{7r+3}{2}=3r+$

$$1+\frac{r+1}{2}=wh.$$

Again, let there be taken $\frac{r+1}{2}=s$, or $r=2s-1$;

Then $q=\frac{7(2s-1)+3}{2}=7s-2$, and $p=\frac{9(7s-2)-3}{7}=9s$ -3; whence we have $x=7(9s-3)+5=63s-16$; where s may be taken at pleasure.

If $s=1$, then $x=47$: and if $s=2$, $x=110$, &c.

Ex. 5. Here by the question we have

$$\frac{x-16}{39}, \text{ and } \frac{x-27}{56} = \text{whole numbers};$$

$$\text{Make } \frac{x-16}{39}=p, \text{ or } x=39p+16.$$

This, substituted in the second equation, gives

$$\frac{39p+16-27}{56}=\frac{39p-11}{56}=p-\frac{17p+11}{56}=wh.$$

$$\text{Make } \frac{17p+11}{56}=q, \text{ or } p=\frac{56q-11}{17}=3q-1+\frac{5q+6}{17};$$

$$\text{Then } \frac{5q+6}{17}=wh.$$

$$\text{Also make } \frac{5q+6}{17}=r;$$

$$\text{Then } q=\frac{17r-6}{5}=3r-1+\frac{2r-1}{5}=wh.$$

$$\text{Make } \frac{2r-1}{5}=s; \text{ then } r=\frac{5s+1}{2}=2s+\frac{s+1}{2}=wh.$$

Let therefore $\frac{s+1}{2}=t$, and we have $s=2t-1$;

Consequently $r=\frac{5(2t-1)+1}{2}=5t-2$, and $q=$

$\frac{17(5t-2)-6}{5}=17t-8$; $p=\frac{56(17t-8)-11}{17}=$

$56t-27$; and $x=39(56t-27)+16=2184t-1037$;

Where t may be assumed at pleasure.

When $t=1$, then $x=1147$, the least number agreeing with these conditions.

Ex. 6. Here putting x for the number sought, we must have

$$\frac{x-5}{7}, \frac{x-7}{8}, \text{ and } \frac{x-8}{9}, \text{ all whole numbers}:$$

From the first, by putting $\frac{x-5}{7}=p$, we obtain $x=7p+5$; and this substituted in the other two, gives

$$\frac{7p-2}{8} \text{ and } \frac{7p-3}{9} = \text{whole numbers:}$$

Make $\frac{7p-2}{8}=q$; then $p=\frac{8q+2}{7}=q+\frac{q+2}{7}=wh.$

Let now $\frac{q+2}{7}=r$, and we have $q=7r-2$;

Whence again $p=\frac{8(7r-2)+2}{7}=8r-2$;

Now, substituting this value of p in the third equation, we shall have

$$\frac{7(8r-2)-3}{9}=\frac{56r-17}{9}= \text{a whole number;}$$

$$\text{or } 6r-2+\frac{2r+1}{9}=wh.$$

Let therefore $\frac{2r+1}{9}=s$, or $r=\frac{9s-1}{2}=4s+\frac{s-1}{2}$,

Where $\frac{s-1}{2}$ is likewise $=wh.$

And let $\frac{s-1}{2}=t$, and $s=2t+1$.

From this we readily draw the following values,

$$s=2t+1;\ r=\frac{9(2t+1)-1}{2}=9t+4,$$

$$q=7(9t+4)-2=63t+26,$$

$$p=\frac{8(63t+26)+2}{7}=72t+30, \text{ and}$$

$$x=7(72t+30)+5=504t+215,$$

Where t may be assumed at pleasure.

If $t=0$, then $x=215$; If $t=2$, then $x=1223$;
If $t=1$, then $x=719$; If $t=3$, then $x=1727$, &c.

Ex. 7. Let x be the number sought; then, by the question,

$\frac{x}{9}, \frac{x}{8}, \frac{x}{7}, \frac{x}{6}, \frac{x}{5}, \frac{x}{4}, \frac{x}{3}$, and $\frac{x}{2}$, must all be whole numbers.

Now, first, if $\frac{x}{9}$ and $\frac{x}{8}$ be whole numbers, $\frac{x}{6}, \frac{x}{4}, \frac{x}{3}$ and $\frac{x}{2}$ must necessarily be so.

We have, therefore, in the present case, only to find

$$\frac{x}{9}, \frac{x}{8}, \frac{x}{7}, \frac{x}{5}$$

whole numbers, which must have place, if x be made equal to 2520, the product of all these denominators.

Ex. 8. Here, if we put x for the number, the conditions are, that

$$\frac{x}{2}, \frac{x}{3}, \frac{x}{4}, \frac{x}{5}, \frac{x}{6}, \text{ and } \frac{x-5}{7}$$

must all be integers, or whole numbers.

But the first five of these fractions, when brought to a common denominator, are

$$\frac{30x}{60}, \frac{20x}{60}, \frac{15x}{60}, \frac{12x}{60}, \text{ and } \frac{10x}{60};$$

Whence, as any multiple of a whole number is a whole number, we have only to make $\frac{x}{60}$ and $\frac{x-5}{7}=$ whole numbers.

Or, as in the preceding examples, these conditions may be reduced, by observing that if $\frac{x}{2}, \frac{x}{5}$, and $\frac{x}{6}$ are whole numbers, $\frac{x}{3}$ and $\frac{x}{4}$ must be so likewise;

And the least value of x, which answers the three former conditions, is $x=2\times5\times6=60$; x, therefore, must be some multiple of 60.

Let then $x=60p$, and the last condition is that $\frac{60p-5}{7}=$ *whole number*.

$$\text{Now } \frac{60p-5}{7}=9p-1-\frac{3p-2}{7};$$

$$\text{Therefore } \frac{3p-2}{7}= \textit{whole number}.$$

Make $\frac{3p-2}{7}=q$, then $p=\frac{7q+2}{3}=2q+1+\frac{q-1}{3}$:

Again, let $\frac{q-1}{3}=r$, and we have $q=3r+1$; and consequently $p=\frac{7(3r+1)+2}{3}=7r+3$.

Where r may be assumed $=0$, or any integer whatever.
If $r=0$, then $p=3$, and $x=60p=180$.

DIOPHANTINE ANALYSIS.

QUESTIONS FOR PRACTICE.

Ex. 1. Since $x+1$, and $x-1$, are to be both squares, let $x=y^2-1$, then $x+1=y^2$, which fulfils the first condition; and therefore it only remains to make

$$y^2-2=\square.$$

Let $y^2-2=(y-1)^2=y^2-2y+1$, and we shall have $2y=3$, or $y=\frac{3}{2}$, or $y^2=\frac{9}{4}$;
Consequently $x=y^2-1=\frac{9}{4}-1=\frac{5}{4}$, the number sought.

Ex. 2. Here $x+4$, and $x+7$, must become squares.
Let $x+4=p^2$, or $x=p^2-4$, then $x+7=p^2+3=\square$.
Let $p^2+3=(p+q)^2=p^2+2pq+q^2$;
Whence $p=\frac{3-q^2}{2q}$, and, consequently, $x=\frac{9-22q^2+q^4}{4q^2}$:
and if we take $\frac{r}{s}=q$, we find $x=\frac{9s^4-22r^2s^2+r^4}{4r^2s^2}$, in which we may substitute for r and s any integer numbers what-

swer. Let s be taken $=2$, and $r=1$, we have $x=\frac{57}{16}$, whence $x+4=\frac{121}{16}$, and $x+7=\frac{169}{16}$.

Ex. 3. Here $10+x$, and $10-x$, must become squares. This might be done by the method that has just been employed; but let us explain another mode of proceeding.

It is evident that $(10+x)\times(10-x)=100-x^2$, must likewise become a square; now its first term being already a square, let $100-x^2=(10-px)^2=100-20px+p^2x^2$; therefore $x=\frac{20p}{p^2+1}$;

Now it is only necessary to make one of these formulae $=\square$, because it necessarily follows that the other will be a square; now $10+x=\frac{10p^2+20p+10}{p^2+1}=\frac{10(p^2+2p+1)}{p^2+1}$; and since $p^2+2p+1=\square$, $\frac{10}{p+1}$, or $\frac{10p^2+10}{(p^2+1)^2}=\square$; but the denominator $=\square$, hence $10p^2+10=\square$; and here it is necessary to find a case in which that takes place.

When $p=3$, then $10p^2+10=90+10=100=\square$. Let $p=3+q$, and we shall have $100+60q+10q^2=\square$. Let the root of this be $10+qt$, and we shall have $100+60q+10q^2=100+20qt+q^2t^2$, whence $q=\frac{60-20t}{t^2-10}$, by which means we shall determine $p=3+q$, and $x=\frac{20p}{p^2+1}$. Let $t=3$, then $q=0$, and $p=3$; therefore $x=6$, and, consequently, $10+x=16$, and $10-x=4$.*

Ex. 4. Here x^2+1, and $x+1$, are to be both squares.

Let $x+1=p^2$, or $x=p^2-1$; then $x^2+1=p^4-2p^2+2=\square$; which last formula is of such a nature as not to admit of a solution, unless we already know a satisfactory case; but this is a square when $p=1$; therefore let $p=1+q$, and we

* See, for a more general solution of this and several of the following questions, Euler's Algebra, vol. ii. p. 174.

shall have $x^2+1=1+4q^2+4q^3+q^4$, which may become a square several ways. Let $1+2q^2=$ the side of this square; then $1+4q^2+4q^3+q^4=1+4q^2+4q^4$: therefore $q=\frac{4}{3}$, and $p=\frac{7}{3}$; consequently $x=\frac{40}{9}$.

Ex. 5. Let x^2, y^2, and z^2, be the numbers sought.

Then $x^2+z^2=\square$, $y^2+z^2=\square$, and $x^2+y^2=\square$.

Or, $\frac{x^2}{z^2}+1=\square$, $\frac{y^2}{z^2}+1=\square$, and $\frac{x^2}{z^2}+\frac{y^2}{z^2}=\square$.

And, by putting $\frac{x}{z}=\frac{s^2-1}{2s}$, and $\frac{y}{z}=\frac{r^2-1}{2r}$, we shall have $\frac{x^2}{z^2}+1=\frac{s^4+2s^2+1}{4s^2}$, and $\frac{y^2}{z^2}+1=\frac{r^4+2r^2+1}{4r^2}$, which are both evidently squares; and therefore it only remains to make $\frac{x^2}{z^2}+\frac{y^2}{z^2}=$ a square.

But $\frac{x^2}{z^2}+\frac{y^2}{z^2}=(\frac{s^2-1}{2s})^2+(\frac{r^2-1}{2r})^2=\frac{(s^2-1)^2}{4s^2}+\frac{(r^2-1)^2}{4r^2}$ $=\frac{4r^2(s^2-1)^2+4s^2(r^2-1)^2}{4r^2s^2}=$ to a square number.

Or, $r^2\times(s^2-1)^2+s^2(r^2-1)^2=r^2(s+1)^2\times(s-1)^2+s^2\cdot(r+1)^2\times(r-1)^2=$ to a square number. And, by making $r-1=s+1$, or $r=s+2$, we shall have $(s+2)^2\times(s+1)^2\times(s-1)^2+s^2\times(s+3)^2\times(s+1)^2=$ to a square number. Or, $(s+2)^2\times(s-1)^2+s^2\times(s+3)^3=2s^4+8s^3+6s^2-4s+4=$ to a square number.

Let, now, the root of this last square be assumed $=\frac{5}{4}s^2-s+2$: then, $2s^4+8s^3+6s^2-4s+4=(\frac{5}{4}s^2-s+2)^2=\frac{25}{16}s^4-\frac{5}{2}s^3+5s^2+s^2-4s+4$; or $2s^4+8s^3=\frac{25}{16}s^4-\frac{5}{2}s^3$; or, $2s+8=\frac{25}{16}s-\frac{5}{2}$.

Whence $s=-24$, and $r=-22$. And $\frac{x}{z}=\frac{s^2-1}{2s}=-\frac{575}{48}$, and $\frac{y}{z}=\frac{r^2-1}{2r}=-\frac{483}{44}$; or $x=-\frac{575z}{48}$, and $y=-\frac{483z}{44}$.

In order, therefore, to have the answer in whole numbers, let $z=528$, and we shall have $x=-6325$, and $y=-5796$. And consequently, $(528)^2$, $(5796)^2$, and $(6325)^2$, are the squares required.

Ex. 6. Let the numbers sought be x and y; then $x^2+y=\square$, and $y^2+x=\square$. And, if $r-x$ be assumed for the side of the first square x^2+y, we shall have $x^2+y=r^2-2rx+x^2$, or $y=r^2-2rx$. Therefore, by reduction, $2rx=r^2-y$, or $x=\frac{r^2-y}{2r}$. Again, if $y+s$ be assumed for the side of the second square, we shall have $y^2+\frac{r^2-y}{2r}=(y+s)^2=y^2+2sy+s^2$. Whence also $\frac{r^2-y}{2r}=2sy+s^2$, or $r^2-y=4rsy+2rs^2$. Whence, by transposition and division, we have $y=\frac{r^2-2rs^2}{4rs+1}$, and $x=\frac{r^2-y}{2r}=\frac{2r^2s+s^2}{4rs+1}$; where r and s may be taken at pleasure, provided r be greater than $2s^2$.

But to find the answer in the book, according to Euler's method, let $x^2+y=(p-x)^2$; and, at the same time, the other $y^2+x=(q-y)^2$, and we shall thus obtain the two following equations, $y+2px=p^2$, and $x+2qy=q^2$, from which we easily deduce $x=\frac{2qp^2-q^2}{4pq-1}$, and $y=\frac{2pq^2-p^2}{4pq-1}$, in which p and q are indeterminate.

Let $p=1$, and $q=\frac{3}{2}$, we shall then have $x=\frac{3}{20}$, and $y=\frac{7}{10}$, whence we derive $x^2+y=\frac{9}{400}+\frac{7}{10}=\frac{289}{400}=(\frac{17}{20})^2$, and $y^2+x=\frac{49}{100}+\frac{3}{20}=\frac{64}{100}=(\frac{8}{10})^2$.

Ex. 7. Let x^2, y^2, and z^2, be the numbers required; then, by the nature of harmonical proportion, $x^2-y^2 : y^2-z^2 :: x^2 : z^2$; hence $y^2=\frac{2x^2z^2}{x^2+z^2}=\frac{x^2z^2}{\frac{1}{2}(x^2+z^2)}=\frac{x^2z^2}{(\frac{x+z}{2})^2+(\frac{x-z}{2})^2}=$ a square; therefore, since the nume-

rator is already a square,) $(\frac{x+z}{2})^2+(\frac{x-z}{2})^2$ must be a square; we have, therefore, to find two squares $(\frac{x+z}{2})^2$ and $(\frac{x-z}{2})^2$, such, that their sum may be a square. This will be accomplished by taking $\frac{x+z}{2}=2rs$, and $\frac{x-z}{2}=r^2-s^2$; for then $y=r^2+s^2$, as appears from note, page 228 in the Introduction. Hence, $x=\frac{x+z}{2}+\frac{x-z}{2}=2rs-r^2+s^2$; and $z=\frac{x+z}{2}-\frac{x-z}{2}=2rs+r^2-s^2$; then, y, or the root of $\frac{2x^2z^2}{x^2+z^2}$, will be $=\frac{6r^2s^2-r^4-s^4}{r^2+s^2}$. Therefore, $2rs-r^2+s^2$, $\frac{6r^2s^2-r^4-s^4}{r^2+s^2}$, and $2rs+r^2-s^2$, will be the roots of three squares in harmonical proportion. Or, if each number be multiplied by r^2+s^2, we shall have $2r^3s+2rs^3-r^4+s^4$, $6r^2s^2-r^4-s^4$, and $2r^3s+2rs^3+r^4-s^4$, for the three roots, where r and s may be taken at pleasure.

If $r=1$, and $s=2$; then $x=35$, $y=7$, and $z=5$; and the three squares are 35^2, 7^2, and 5^2, or 1225, 49, and 25.

Ex. 8. Let x^3, y^3, and z^3, be three cubes, such, that their sum $x^3+y^3+z^3$, may be equal to a cube $=v^3$; then, by transposing one of the terms, we have $x^3+y^3=v^3-z^3$. In order to satisfy the conditions of this equation, put $x=u+w$, and $y=u-w$; then we have $x^3+y^3=2u(u^2+3w^2)$. Also, put $v=r+s$, and $z=r-s$, and then we have $v^3-z^3=2s(s^2+3r^2)$; whence $2u(u^2+3w^2)=2s(s^2+3r^2)$, or $u(u^2+3w^2)=s(s^2+3r^2)$.

Assume $u=mt+3np$, and $w=nt-mp$; then $u(u^2+3w^2)=(mt+3np)(m^2+3n^2)(t^2+3p^2)$. Also, assume $s=at+3cp$, and $r=ct-ap$; then $s(s^2+3r^2)=(at+3cp)(a^2+3c^2)(t^2+3p^2)$, whence $(mt+3np)(m^2+3n^2)(t^2+3p^2)=(at+3cp)(a^2+3c^2)(t^2+3p^2)$; or dividing by t^2+3p^2, $(mt+3np)(m^2+3n^2)=(at+3cp)(a^2+3c^2)$, and $t=\frac{3c(a^2+3c^2)-3n(m^2+3n^2)}{m(m^2+3n^2)-a(a^2+3c^2)}p$.

Or, if p be taken $=m(m^2+3n^2)-a(a^2+3c^2)$, then $t=3c(a^2+3c^2)-3n(m^2+3n^2)$; where a, c, m, and n, may be any numbers taken at pleasure. If we suppose $n=0$, and $a=c$, then $t=12c^3$, and $p=m^3-4c^3$; hence $u=12mc^3$, $w=4mc^3-m^4$, $s=3cm^3$, and $r=16c^4-cm^3$; therefore $x=u+w=16mc^3-m^4$, $y=u-w=8mc^3+m^4$, $z=r-s=16c^4-4cm^3$, and $v=r+s=16c^4+12cm^3$. Now, if $c=m=1$, then $x=15$, $y=9$, $z=12$, and $v=18$; or dividing by 3, $x=5$, $y=3$, $z=4$, and $v=6$: so that $3^3+4^3+5^3=6^3$.

Again, if we suppose $a=0$, and $c=1$, we shall find $x=-3n-(m^2+3n^2)+m$, $y=-3n+(m^2+3n^2)-m$, $z=(3n-m)(m^2+3n^2)+1$, and $v=-(3n+m)(m^2+3n^2)+1$; where, if $m=-1$, and $n=1$, then $x=-20$, $y=14$, $z=17$, and $v=-7$, so that we have $14^3+17^3+7^3=20^3$.

Ex. 9. Let x be one of the parts, then $100-x^2$ will be the other part, which is also a square number. Assume the side of this second square $=2x-10$,* then will $100-x^2=(2x-10)^2=4x^2-40x+100$; and, consequently, by reduction, $x=8$, and $2x-10=6$. Therefore 64 and 36 are the parts required.

Ex. 10. Let x and y denote the numbers sought; then, by the question, $x^2-y^2=x-y$, and $x^2+y^2=\square$. The first equation, divided by $x-y$, gives $x+y=1$; hence $x=1-y$, and $x^2+y^2=1-2y+2y^2=$ a square.

Assume $1-ry$ for its side: then $1-2y+2y^2=1-2ry+r^2y^2$; whence $y=\frac{2r-2}{r^2-2}$, and $x=1-y=\frac{r^2-2r}{r^2-2}$; where r may be taken at pleasure, provided it be greater than 2.

If $r=3$, then $y=\frac{4}{7}$, and $x=\frac{3}{7}$, the answer.

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* If $x-10$ had been made the side of the second square, in the above solution of this question, instead of $2x-10$, the equation would have been $x^2-20x+100=100-x^2$; in which case, x the side of the first square, would have been found $=10$, and $x-10$, or the side of the second square $=0$; for which reason the substitution $x-10$ was avoided; but $3x-10$, $4x-10$, or any other quantity of the same kind, would have succeeded as well as the former, though the results would have been less simple. This question is done generally, in example 4, preceding questions for practice in the Introduction.

Ex. 11. Here, if we call x and y the two numbers, we have to find
$$x+xy=\square$$
$$y+xy=\square.$$

Now if we assume $x=m^2$, and $y=n^2$, these become
$$m^2+m^2n^2=m^2(1+n^2)$$
$$n^2+n^2m^2=n^2(1+m^2)$$
which must be both squares; that is, we must have $1+n^2$, and $1+m^2$, both squares.

Assume $n^2+1=(n+r)^2=n^2+2rn+r^2$;

Then, from this we have $n=\frac{1-r^2}{2r}$; and in the same manner we have $m=\frac{1-s^2}{2s}$;

Consequently $x=(\frac{1-r^2}{2r})^2$, and $y=(\frac{1-s^2}{2s})^2$; where r and s may be assumed at pleasure.

If $r=2$, and $s=3$, then $x=\frac{9}{16}$ and $y=\frac{16}{9}$.

Which numbers answer the conditions of the question; but they are both square numbers, which is not a necessary condition.

The question may, therefore, be otherwise resolved, as follows:

Let x and $x-1$ be the two numbers; then we have to find
$$x+x(x-1)=x^2, \text{ and}$$
$$(x-1)+x(x-1)=x^2-1.$$
both squares. The first of which is so, and therefore it only remains to find x^2-1 a square.

Make $x^2-1=(x-r)^2=x^2-2rx+r^2$,

And we have $x=\frac{r^2+1}{2r}$; where r may be any number greater than 1.

If $r=3$, then $x=\frac{5}{3}$ and $x-1=\frac{2}{3}$, which numbers answer the conditions of the question.

Ex. 12. Let x^2, y^2, and z^2 represent the required squares; and we shall have to solve the equation

$$x^2+z^2=2y^2$$

In order to which, let $x=m+n$, and $z=m-n$, then

$$x^2+z^2=2m^2+2n^2=2y^2;$$

In which case, it only remains to find

$$m^2+n^2=y^2.$$

Let $m=p^2-q^2$, and $n=2pq$, and we have at once

$$m^2+n^2=(p^2-q^2)^2+4p^2q^2=(p^2+q^2)^2=y^2.$$

Hence the following general results, viz.

$$x=p^2-q^2+2pq, \text{ and } x^2=(p^2-q^2+2pq)^2$$
$$y=p^2+q^2 \qquad x^2=(p^2+q^2)^2$$
$$z=p^2-q^2-2pq \qquad z^2=(p^2-q^2-2pq)^2$$

Where p and q may be assumed at pleasure.

If $p=2$, and $q=1$, then $x=7$, $y=5$, and $z=-1$;
Consequently, the squares are 49, 25, and 1.

Ex. 13. Let $\frac{1}{2}x^2-y$, $\frac{1}{2}x^2$, and $\frac{1}{2}x^2+y$, be the three numbers in arithmetical progression;

Then, we have to find x^2, x^2+y, and x^2-y, rational squares; or $x^2+y=m^2$, and $x^2-y=n^2$

Assume $y=2rx+r^2$, then we have

$$x^2+y=x^2+2rx+r^2=(x+r)^2$$

And therefore it only remains to find

$$x^2-2rx-r^2=n^2.$$

Assume $x^2-2rx-r^2=(x-m)^2=x^2-2mx+m^2$

And we shall have $x=\dfrac{m^2+r^2}{2m-2r}$,

Where m and r may be taken at pleasure.

If $m=5$, and $r=4$; then, $x=\frac{41}{2}$, and $\frac{1}{2}x^2=\frac{1681}{8}$.

Also, $y=2rx+r^2=41\times4+16=180$; whence, the three numbers will be $30\frac{1}{8}$, $210\frac{1}{8}$, and $390\frac{1}{8}$.

And if these numbers be multiplied by any square number, the same conditions will obviously obtain:

Hence, multiplying by 4, we shall have

$$120\tfrac{1}{2},\ 840\tfrac{1}{2},\ \text{and } 1560\tfrac{1}{2};$$

which are the numbers given in the answer in the Introduction.

And if these last be again multiplied by 4, we shall have 482, 3362, 6242, which are all integral, and equally answer the conditions of the question.

The question may be otherwise solved as follows:

Let x, y, and z, be the numbers, and assume

$$x+y=m^2$$
$$x+z=n^2$$
$$y+z=r^2$$

Then we shall have

$$x=\tfrac{1}{2}(\ \ m^2+n^2-r^2)$$
$$y=\tfrac{1}{2}(\ \ m^2-n^2+r^2)$$
$$z=\tfrac{1}{2}(-m^2+n^2+r^2)$$

Where, since the numbers are in arithmetical progression, we have

$$x+z=2y\ ;\ \text{or}\ n^2=m^2-n^2+r^2:$$

Therefore we have only to find $m^2+r^2=2n^2$; that is, three square numbers in arithmetical progression:

Whence, from Example 12, we shall have

$$m^2=(p^2-q^2-2pq)^2$$
$$n^2=(p^2+q^2)^2$$
$$r^2=(p^2-q^2+2pq)^2$$

And consequently

$$x=\tfrac{1}{2}(p^2+q^2)^2-4pq(p^2-q^2)$$
$$y=\tfrac{1}{2}(p^2+q^2)^2$$
$$z=\tfrac{1}{2}(p^2+q^2)^2+4pq(p^2-q^2)$$

Where p and q may be taken at pleasure.

Ex. 14. Let x, y, and z, be the numbers sought; then by the question,

$$x^2+y+z=m^2$$
$$y^2+x+z=q^2$$
$$z^2+x+y=r^2$$

Assume $x^2+y+z=(x+n)^2=x^2+2nx+n^2$:

Then we shall have $x=\dfrac{y+z-n^2}{2n}$:

This value of x, substituted in our second and third equations, gives

$$y^2+\frac{y+z-n^2}{2n}+z=p^2$$

$$z^2+\frac{y+z-n^2}{2n}+y=r^2$$

Hence, assume $y^2+\frac{y+z-n^2}{2n}+z=(y-p)^2=y^2-2py+p^2;$

And $z^2+\frac{y+z-n^2}{2n}+y=(z-s)^2=z^2-2sz+s^2$

And we shall now obtain

From the 1st. $z=\frac{n^2-y-4pny+2np^2}{1+2n}$,

From the 2d. $z=\frac{n^2-y-2ny+2ns^2}{1+4ns}$;

Whence,

$$\frac{n^2-y-4pny+2np^2}{1+2n}=\frac{n^2-y-2ny+2ns^2}{1+4ns},$$

And consequently by reduction,

$$y=\frac{4nsp^2+2n^2s+p^2-n^2-s^2-2ns^2}{2p+2s+8nps-2-2n}$$

Where n, p, and s, may be assumed at pleasure.

If $n=1$, $p=4$, and $s=11$, then $y=1$, $z=\frac{16}{3}$, and $x=\frac{8}{3}$; which are the answers given in the Introduction.

Ex. 15. Assume $8x$ and $15x$ for the two numbers; then, by the question, $(8x)^2+(15x)^2$ is to be a square.

But $(8x)^2+(15x)^2=289x^2=(17x)^2$; therefore, x may be any number at pleasure. If $x=72$, we have 576 and 1080 for the numbers sought.

Ex. 16. Let x and y be the numbers sought; then, by the question, we have to solve the equations

$$x^2+xy, \text{ or } x(x+y)=m^2$$
$$y^2+xy, \text{ or } y(x+y)=n^2$$

Assume $x=p^2$, and $y=q^2$, and these become

$$p^2(p^2+q^2)=m^2$$
$$q^2(p^2+q^2)=n^2$$

Which conditions will be fulfilled, if we find $p^2+q^2=$ a square.

Now, we have already seen (Example 12) that for this purpose we have only to assume

$$p=r^2-s^2, \text{ and } q=2rs$$
$$\text{Therefore, } x=p^2=(r^2-s^2)^2$$
$$\text{And } y=q^2=4r^2s^2$$

Where r and s may be assumed at pleasure.

If $r=2$, and $s=1$, then $x=9$, and $y=16$. These, however, are square numbers, which is not a necessary condition.

Now, it is obvious that any multiple whatever of the same numbers will equally answer the purposes required in the question.

We shall have, therefore, a more general solution by taking $x=t(r^2-s^2)^2$, and $y=4tr^2s^2$; where r, s, and t may be assumed at pleasure.

Ex. 17. Let x and y represent the two numbers; then, by the question, $x^2+y^2-1=m^2$

$$x^2-y^2-1=n^2$$

Assume $x=y+1$; then, these two equations reduce to

$$(y+1)^2+y^2-1=2y^2+2y=m^2$$
$$(y+1)^2-y^2-1=\qquad 2y=n^2$$

Where, putting the first under the form $2y(y+1)=m^2$, and substituting $2y=n^2$, this becomes

$$n^2(y+1)=m^2,$$

Consequently, $y+1$ must be a square; let, therefore, $y+1=p^2$; then, since we have found $2y=n^2$, we have, by subtraction $y=n^2-p^2+1$

And, consequently, $x=n^2-p^2+2$

Where n and p may be assumed at pleasure. If $n=4$, and $p=3$, then $y=8$, and $x=9$, the numbers required.

Ex. 18. Without attending to the particular numbers in the question, let us endeavour to resolve any given square number into two other square numbers.

For this purpose, let a^2 represent the given square that is to be so resolved, and put x^2 and y^2 for the required squares.

Then we have to satisfy the equation

$$a^2=x^2+y^2,$$

$$\text{or } a^2-y^2=x^2.$$

In order to which, let us assume

$$a+y=\frac{px}{q}$$

$$a-y=\frac{qx}{p}$$

From which we have, by addition and subtraction,

$$2a=\frac{px}{q}+\frac{qx}{p}=\frac{(p^2+q^2)x}{pq}$$

$$2y=\frac{px}{q}-\frac{qx}{p}=\frac{(p^2-q^2)x}{pq}$$

Whence, by multiplication and division,

$$x=\frac{2pqa}{p^2+q^2}$$

$$y=\frac{(p^2-q^2)a}{p^2+p^2}$$

Where the indeterminates p and q may be assumed at pleasure.

But it is obvious that $2pq$ and p^2+q^2, as also p^2-q^2 and p^2+q^2, being incommensurate, the values of x and y must be fractional, except a be divisible by p^2+q^2.

That is, unless a be divisible by the sum of two squares; and then the question will admit of as many integral answers as a has divisors of this form.

Now, by the question, $a=65$; and 65 is divisible by $5=2^2+1^2$, by $13=3^2+2^2$, and by $65=7^2+4^2=8^2+1^2$.

Hence, we may assume $p=2$, and $q=1$; or $p=3$, and $q=2$; or $p=7$, and $q=4$; or $p=8$, and $q=1$;

Which give the four following solutions, $65^2=16^2+63^2=56^2+33^2=60^2+25^2=52^2+39^2$.

These being the only integral answers the question admits of.

Ex. 19. Let x^2, y, and $\frac{y^2}{x^2}$, be the numbers required, and put the given number $19=a$; then, by the question,

x^2+a, $y+a$, and $\frac{y^2}{x^2}+a$, are to be all squares, or

$$x^2+a=m^2$$
$$y+a=n^2$$
$$\frac{y^2}{x^2}+a=v^2$$

Assume $x^2+a=(r-x)^2=r^2-2rx+x^2$,
and $y+a=r^2$;

Then, we have $x=\frac{r^2-a}{2r}$, and $y=r^2-a$;

Wherefore, $\frac{x}{y}=2r$, and $\frac{y^2}{x^2}+a=4r^2+a$, which must be a square.

Let, therefore, $4r^2+a=(2r+s)^2=4r^2+4rs+s^2$;

In which case, we shall have $r=\frac{a-s^2}{4s}$.

And, consequently, $y=(\frac{a-s^2}{4s})^2-a$

Where s may be taken at pleasure, provided $(\frac{a-s^2}{4s})^2$ be greater than a.

But, by the question, $a=19$; if, therefore, we assume $s=1$, we have $y=(\frac{19-1}{4})^2-19=\frac{5}{4}$; $\frac{y}{x}=\frac{a-s^2}{2s}=\frac{19-1}{2}=9$,

and $x=\frac{5}{4}\div 9=\frac{5}{36}$.

Hence, the three numbers are

$$x^2=(\frac{5}{36})^2=\frac{25}{1296}, \; y=\frac{5}{4}, \text{ and } \frac{y^2}{x^2}=9^2=81.$$

The question may be otherwise answered, thus:

Let $4x^2$, and x^2-a, be two of the numbers; then $\frac{(x^2-a)^2}{4x^4}$

will be the third number; and, by the question,

$$4x^2+a=m^2$$
$$x^2-a+a=n^2$$
$$\frac{(x^2-a)^2}{4x^2}+a=(x^2+a)^2=v^2.$$

Here, the second and third equations being squares, it only remains to find the first $4x^2+a=m^2$.

Assume $4x^2+a=(2x+s)^2=4x^2+4sx+s^2$,

Whence, $x=\frac{a-s^2}{4s}$; where s may be taken at pleasure, provided only that s^2 be less than a.

Ex. 20. Let x and y be the two numbers we have to find; then, $x^2+y^2+xy=m^2$

Assume $x^2+xy+y^2=(x+r)^2=x^2+2rx+r^2$

And we shall have $x=\frac{y^2-r^2}{2r-y}$.

Where r and y may be taken at pleasure, provided y be greater than r, but less than $2r$.

If $y=3$, and $r=2$, then $x=5$
$y=5$, and $r=3$, then $x=16$
$y=7$, and $r=5$, then $x=8$.

Ex. 21. If x, y, and z are taken for the three numbers, we have to find

$$x^2+zy=\square$$
$$y^2+xz=\square$$
$$z^2+xy=\square$$

Or, dividing by x^2, we have, in that case,

$$1+\frac{zy}{x^2}=\square$$
$$\frac{y^2}{x^2}+\frac{z}{x}=\square$$
$$\frac{z^2}{x^2}+\frac{y}{x}=\square$$

Or if, in order to simplify the expression, we put $\frac{y}{z}=m$, and $\frac{z}{x}=n$, the above will become

$$mn+1=\square=r^2$$
$$m^2+n=\square=s^2$$
$$n^2+m=\square=t^2$$

From the first, $mn=r^2-1$; assume, therefore,

$$m=r+1$$
$$n=r-1$$

And there now remains to find

$$(r+1)^2+(r-1)=s^2$$
$$(r-1)^2+(r+1)=t^2;\text{ or}$$
$$r^2+3r=s^2$$
$$r^2-r+2=t^2$$

Assume $r^2+3r=(r+w)^2=r^2+2rw+w^2$

Whence, $r=\frac{w^2}{3-2w}$; which, substituted, gives

$$\frac{w^4}{(3-2w)^2}-\frac{w^2(3-2w)}{(3-2w)^2}+\frac{2(3-2w)^2}{(3-2w)^2}=t^2$$

Or $w^4+2w^3+5w^2-12w+18=\square$

Assume this $=(w^2+w+2)^2=$

$$w^4+2w^3+5w^2+4w=4,$$

Then, $-12w+18=4w+4$,

$$\text{or } w=\frac{14}{16}=\frac{7}{8}.$$

And we shall have $r=\frac{w^2}{3-2w}=\frac{49}{80}$.

Consequently, $m=\frac{129}{80}$, and $n=\frac{-31}{80}$,

Or $x=80$, $y=129$, and $z=-31$,

Which are three numbers answering the required conditions, and a similar process will give the three numbers, —9, 73, 328, all positive.

Ex. 22. Find three squares, a^2, b^2, c^2, such, that their sum may be a square,

$$\text{or } a^2+b^2+c^2=d^2$$

That is, assume $a^2+b^2+c^2=(c+r)^2$; from which we

$$\text{shall obtain } c=\frac{a^2+b^2-r^2}{2r},$$

where a, b, and r, may be taken at pleasure, provided r^2 be less than a^2+b^2.

This being done, let ax, bx, and cx, be the required squares; then, $a^2x^2+b^2x^2+c^2x^2=d^2x^2$.

And we have to find

$$d^2x^2+ax=\Box$$
$$d^2x^2+bx=\Box$$
$$d^2x^2+cx=\Box$$

Or, dividing by d^2, and putting $\frac{a}{d^2}=m$, $\frac{b}{d^2}=n$, and $\frac{c}{d^2}=p$; these become

$$x^2+mx=\Box$$
$$x^2+nx=\Box$$
$$x^2+px=\Box$$

Assume $x^2+mx=(r-x)^2=r^2-2rx+x^2$

$$\text{Then we have } x=\frac{r^2}{2r+m}.$$

This value of x, substituted in the second and third, gives

$$\frac{r^2}{(2r+m)^2}\times\{r^2+n(2r+m)\}=\Box$$
$$\frac{r^2}{(2r+m)^2}\times\{r^2+p(2r+m)\}=\Box;$$

And, since the first two factors of each are squares, we have only to find

$$r^2+n(2r+m)=v^2$$
$$r^2+p(2r+m)=w^2$$

$$\text{Make } r^2+n(2r+m)=(s-r)^2=s^2-2sr+r^2$$

$$\text{Then we have } r=\frac{s^2-nm}{2(n+s)}.$$

Which value of r, substituted in the latter equation, gives

$$\left[\frac{s^2-nm}{2(n+s)}\right]^2+p\left(\frac{s^2-mn}{n+s}+m\right)=w^2,$$

Which, by reduction, becomes

$$\frac{(s^2-nm)^2+4ps(s+m)(n+s)}{4(n+s)^2}=w^2$$

Or $(s^2-nm)^2+4ps(s+m)(s+n)=w^2$,

Assume this $=[(s^2-nm)^2-2ps]^2=$

$(s^2-nm)^2-4ps(s^2-nm)+4p^2s^2$

Then, this, by reduction, gives

$(s+m)(s+n)=nm-s^2+ps$

Whence, again, $s=\dfrac{p-(m+n)}{2}$,

$$\text{And } x=\frac{r^2}{2r+m}=\frac{(s^2-nm)^2(n+s)}{4(s^2+ms)(n+s)^2}=\frac{(s^2-nm)^2}{4s(s+m)(s+n)}.$$

Now, if we take $a=2$, $b=6$, and $c=9$, then $d=11$,

$m=\dfrac{a}{d^2}=\dfrac{2}{121}$, $n=\dfrac{b}{d^2}=\dfrac{6}{121}$, and $p=\dfrac{c}{d^2}=\dfrac{9}{121}$.

$s=\dfrac{p-(m+n)}{2}=\dfrac{9-8}{121}\div 2=\dfrac{1}{242}$, and $x=\dfrac{2209}{62920}$,

Whence, $ax=2x=\dfrac{4418}{62920}$

$bx=6x=\dfrac{13254}{62920}$

$cx=9x=\dfrac{19881}{62920}$

Ex. 23. Let x, rx, and r^2x be the three numbers in geometrical progression; then, by the question,

$$x+rx=x(1+r)=v^2$$
$$rx+r^2x=rx(1+r)=w^2$$

Dividing the second by the first, it is obvious that r

must be a square; let, therefore, $r=y^2$, and it only remains to find $$x(1+y^2)=v^2.$$

Which equation we will obtain, if we make
$$x=nz^2$$
$$\text{and } 1+y^2=nw^2$$
$$\text{Whence, } x=n(z^2-w^2)+y^2+1,$$
Where all the indeterminates may be taken at pleasure.

If $z=w$,
then, $x=y^2+1$
in which, y may be taken at pleasure.

If $y=2$, then $x=5$, and $r=4$; and the three numbers sought are 5, 20, and 80.

Ex. 24. Let x and y represent the two numbers; then, it is required to find
$$x+1=m^2$$
$$x-1=n^2$$
$$x+y+1=r^2$$
$$x-y+1=s^2$$

Now, here, it is obvious that the three squares, r^2, m^2, and s^2, are in arithmetical progression, their common difference being y.

Let us, therefore, represent these squares as in Ex. 12; viz.
$$s^2=(2pq-p^2-2q^2)^2$$
$$m^2=(p^2+2q^2-2pq)^2$$
$$r^2=(p^2+2q^2)^2$$

Then we have for their common difference
$$y=4p^3q-12p^2q^2+8pq^3$$
and all that is required is to find this quantity *plus* 1 a square, or
$$4p^3q-12p^2q^2+8pq^3+1=n^2$$
Assume, therefore, in this case,
$$n=1+4pq^3$$

And we shall have, by squaring and cancelling the like parts,
$$4p^3q-12p^2q^2=16p^2q^6,$$
$$\text{Whence, } p=4q^5+3q,$$
in which expression, q may be assumed at pleasure.

Thus the general values of x and y will be determined; viz. by first making $p=4q^5+3q$, and then

$$x=(p^2+2q^2-2pq)^2-1$$
$$y=(1+4pq^3)^2-1$$

Where, by taking $q=1$, we have $p=7$; whence, $x=1368$, and $y=840$; which numbers answer the conditions of the question; for

$$1368+1=87^2$$
$$840+1=29^2$$
$$1368+840+1=47^2$$
$$1368-840+1=23^2$$

But these are not the numbers in the answer in the Introduction; they are, however, the least integral numbers; and various others may be found by giving different values to q.

Ex. 25. Let x, y, and z be the three numbers, and s their sum; then, we have to find

$$x^2+s=\square,\ x^2-s=\square$$
$$y^2+s=\square,\ y^2-s=\square$$
$$z^2+s=\square,\ z^2-s=\square$$
And $x+y+z=s$.

Now, if we assume for r and s any numbers whatever, and take afterwards $a=r^2-s^2$, $b=2rs+s^2$, and

$$c=r^2+rs+s^2$$

we shall have $c^2=a^2+ab+b^2$; where a, b, and c, will be known numbers, and will possess the following properties, viz.

$(c^2+b^2)^2\pm4abc(a+b)$, a complete square
$(c^2+a^2)^2\pm4abc(a+b)$, a complete square
$[c^2+(a+b)^2]^2\pm4abc(a+b)$, a complete square.

Then, since a, b, and c are here known numbers, and answer the first conditions of the question, it only remains to fulfil the last condition

$$x+y+z=s.$$

In order to effect which, let all the three latter equations be multiplied by x^2, and we shall have

$$x^2(c^2+b^2)^2\pm4abc(a+b)x^2=\square$$
$$x^2(c^2+a^2)^2\pm4abc(a+b)x^2=\square$$
$$x^2[c^2+(a+b)^2]^2\pm4abc(a+b)x^2=\square$$

And it consequently remains to find
$x(c^2+b^2)+x(c^2+a^2)+x[c^2+(a+b)^2]=4abc(a+b)x^2$,

$$\text{Whence, } x=\frac{3c^2+2a^2+2b^2+2ab}{4abc(a+b)}$$

Or, putting $c^2+b^2=m$, $c^2+a^2=n$, $c^2+(a+b)^2=p$ and $4abc(a+b)=q$

$$\text{we shall have } x=\frac{m+n+p}{q}$$

$$\text{And } mx=\frac{m}{q}(m+n+p)$$

$$nx=\frac{n}{q}(m+n+p)$$

$$px=\frac{p}{q}(m+n+p)$$

Which are the three numbers sought.

If $r=2$, and $s=1$; then, $a=3$, $b=5$, and $c=7$. Also $m=c^2+b^2=74$, $n=c^2+a^2=58$, $p=c^2+(a+b)^2=113$, and $q=4abc(a+b)=3360$.

$$\text{Whence, } x=\frac{m+n+p}{q}=245$$

$$\text{Consequently } mx=\frac{245\times 74}{3360}=\frac{518}{96}$$

$$nx=\frac{245\times 58}{3360}=\frac{406}{96}$$

$$px=\frac{245\times 113}{3360}=\frac{791}{96}$$

are the numbers sought; and an indefinite number of other answers may be obtained by assuming other values for r and s.

Ex. 26. Let x^2, y^2, and z^2 be the required squares; then, we have to find

$$x^4+y^4+z^4=m^2$$

Assume $x^4+y^4+z^4=[(x^2+y^2)-z^2]^2=$
$x^4+2x^2y^2+y^4-2z^2(x^2+y^2)+z^4$

Then, $2x^2y^2=2z^2x^2+2z^2y^2$

And consequently $z^2=\frac{x^2y^2}{x^2+y^2}$

Therefore x^2+y^2 must be a square; which it will be if we assume $x=p^2-q^2$, and $y=2pq$; for then

$$x^2+y^2=(p^2-q^2)^2+4p^2q^2=(p^2+q^2)^2$$

Hence the required squares will now be

$$x^2=(p^2-q^2)^2$$
$$y^2=4p^2q^2$$
$$z^2=\frac{(p^2-q^2)^2 4p^2q^2}{(p^2+q^2)^2}$$

Where p and q may be any numbers taken at pleasure. If $p=2$, and $q=1$, then

$$x^2=(p^2-q^2)^2=3^2=9$$
$$y^2=4p^2q^2=4^2=16$$
$$z^2=\frac{(p^2-q^2)^2\times 4p^2q^2}{(p^2+q^2)^2}=\frac{12^2}{5^2}=\frac{144}{25}$$

which numbers answer the required conditions; and various others may be found by giving different values to p and q.

Ex. 27. Here, if x^2, y^2, and z^2, are taken to represent the three squares, we have to find

$$x^2-y^2=\square\text{, or } m^2$$
$$x^2-z^2=\square\text{, or } n^2$$
$$y^2-z^2=\square\text{, or } r^2$$

Hence, if there be assumed, as in Ex. 18,

$$y=\frac{(p^2-q^2)x}{p^2+q^2}\text{, and } z=\frac{(t^2-v^2)x}{t^2+v^2}$$

The first two equations will be resolved; and therefore it only remains to find

$$y^2-z^2\text{, or }\frac{(p^2-q^2)^2}{(p^2+q^2)^2}-\frac{(t^2-v^2)^2}{(t^2+v^2)^2}=r^2$$

Or $(p^2-q^2)^2(t^2+v^2)^2+(p^2+q^2)^2(t^2-v^2)^2=\square$

$$(p^2t^2+p^2v^2-q^2t^2-q^2v^2)^2-(p^2t^2-p^2v^2+q^2t^2-q^2v^2)^2=\square$$
$$2(p^2t^2-q^2v^2)\times 2(p^2v^2-q^2t^2)=\square$$

Assume $p^2=u^2q^2$, and $v^2=w^2t^2$, and this becomes

$$4(u^2q^2t^2-q^2w^2t^2)\times(u^2w^2t^2q^2-q^2t^2);$$

Or dividing by $4q^2t^2$, it now remains to find

$$(u^2-w^2)(u^2w^2-1)=\square.$$

From which condition we get the following partial solution, viz. $w=\frac{4}{5}$, and $u=\frac{13}{4}$;

Whence $p=\frac{13q}{4}$, and $v=\frac{4t}{5}$.

Therefore the required squares are

$$x^2,\ y^2=\left(\frac{p^2-q^2}{p^2+q^2}\right)x^2=\left(\frac{185}{153}\right)^2x^2,$$

$$\text{and } z^2=\left(\frac{t^2-v^2}{t^2+v^2}\right)^2x^2=\left(\frac{41}{9}\right)^2x^2;$$

where x may be any number assumed at pleasure.

If we take $x=153$, we shall have

697^2, 185^2, and 153^2, for the squares required.

Ex. 28. Without attending to the particular cube given in the question, let us endeavour to resolve generally any cube v^3 into three other cubes; viz. let us find such values for x, y, and z, that

$$x^3+y^3+z^3=v^3, \text{ or}$$
$$x^3+y^3=v^3-z^3.$$

For this purpose assume

$$x=u+w, \text{ and } y=u-w,$$
$$v=r+s, \text{ and } z=r-s:$$

Then $\quad x^3+y^3=2u(u^2+3w^2)$

$\quad v^3+z^3=2s(s^2+3r^2)$

Whence $2u(u^2+3w^2)=2s(s^2+3r^2)$

Assume again $\quad u=mt+3np$

$\quad w=nt-mp$;

Then $u(u^2+3w^2)=(mt+3np)(m^2+3n^2)(t^2+3p^2)$

Also assume $\quad s=at+3cp$

$\quad r=ct-ap$;

Then $s(s^2+3r^2)=(at+3cp)(a^2+3c^2)(t^2+3p^2)$

Whence, and from the preceding equation, we have, after dividing by (t^2+3p^2),

$$(mt+3np)(m^2+3n^2)=(at+3cp)(a^2+3c^2)$$

Whence $t=\frac{3c(a^2+3c^2)-3n(m^2+3n^2)}{m(m^2+3n^2)-a(a^2+3c^2)}-p$;

Or taking the denominator $=p$, we shall have

$$t=3c(a^2+3c^2)-3n(m^2+3n^2)$$
$$p=m(m^2+3n^2)-a(a^2+3c^2)$$

Where m, n, a, and c, may be assumed at pleasure. Supposing this assumption to have been made,

$$u=mt+3np$$
$$w=nt-mp$$
$$s=at+3cp$$
$$r=ct-ap$$

are also determined; and consequently also

$$x=u+w,\qquad y=u-w,$$
$$z=r-s,\ \text{and}\ v=r+s:$$

And hence the proposed equation, viz.

$$x^3+y^3+z^3=v^3$$

is resolved: and in order to accommodate it to any particular cube as 8 (the one proposed in the question) we have only to divide the whole by v^3, and to multiply by 8; in which case we have

$$\frac{8x^3}{v^3}+\frac{8y^3}{v^3}+\frac{8z^3}{v^3}=8,\ \text{as required.}$$

If we assume $c=1$, $m=1$, $n=0$, and $a=1$; then $t=12$, and $p=-3$; which values give

$x=15$, $y=9$, $z=12$, and $v=18$;

Whence. $15^3+9^3+12^3=18^3$, and

$5^3+3^3+4^3=6^3$,

or $\frac{5^3}{3^3}+\frac{3^3}{3^3}+\frac{4^3}{3^3}=2^3$;

Consequently $\frac{125}{27}+1+\frac{64}{27}=8$,

Therefore 1, $\frac{64}{27}$, and $\frac{125}{87}$ are the cubes sought.

Ex. 29. Let x, y, and z, be the roots of the required squares; then we have to find

$$x^2+y^2-z^2=r^2$$
$$x^2+z^2-y^2=s^2$$
$$y^2+z^2-x^2=t^2$$

Where, by first assuming

$$x=p^2+q^2$$
$$y=p^2+pq-q^2$$
$$z=p^2-pq-q^2$$

we shall have

$$x^2+y^2-z^2=(p^2-q^2+2pq)^2$$
$$x^2+z^2-y^2=(p^2-q^2-2pq)^2$$

Where the two first conditions being fulfilled, it only remains to make a square of our third equation; which becomes, by substituting for x, y, and z,

$$y^2+z^2-x^2=p^4-4p^2q^2+q^4=t^2$$

In order to reduce this to a more convenient form, let $p=(2+m)q$; then substituting this for p in the above equation, we have

$$y^2+z^2-x^2=q^4(m^4+8m^3+20m^2+16m+1)$$

where we have now to make the latter factor a square.

For which purpose, assume its root $=m^2+am+1$;

Then, by squaring, we have

$$\begin{array}{l} m^4+2am^3+(a^2+2)m^2+\ 2am+1= \\ m^4+\ 8m^3+\ \ 20m^2\ \ \ \ +16\ m+1 \end{array}$$

Or, by making $2a=+16$, there remains

$$+16m^3+66m^2=8m^3+20m^2$$

Whence $46m^2=-8m^3$; wherefore $m=\frac{-23}{4}$,

But $p=(2+m)q$, or $p=\frac{-15}{4}q$; therefore we have $\frac{p}{q}=\frac{15}{-4}$;

Whence, if $p=15$, $q=-4$, we have

$$x=p^2+q^2\qquad\ \ =241$$
$$y=p^2+pq-q^2=149$$
$$z=p^2-pq-q^2=269$$

Consequently 241^2, 149^2, and 269^2, are squares, having the required conditions; and others might be found by giving different values to q and p.

Ex. 30. Let $(1+x)^3$, $(2-x)^3$, and y^3, be the required cubes; and let a be any given number; then by the question

$$\begin{aligned}(1+x)^3-a&=1+3x+3x^2+x^3-a\\(2-x)^3-a&=8-12x+6x^2-x^3-a\\y^3-a&=y^3-a\end{aligned}$$

Whose sum $9-9x+9x^2+y^3-3a$ is to be a square.

Assume it $=(3x-r)^2=9x^2-6rx+r^2$, and we shall have

$$9-9x+y^3-3a=-6rx+r^2;$$

Whence $x=\dfrac{r^2-y^3-9+3a}{6r-9}$,

Where y and r may be assumed at pleasure, provided $6r$ be greater than 9, and r^2 greater than y^3.

In our question $a=1$, and if we take $r=4$, and $y=2$, we shall have $x=\dfrac{2}{15}$;

Whence $\quad (1+x)^3=(1+\dfrac{2}{15})^3=\dfrac{17^3}{15^3}$

$$(2-x)^3=(2-\frac{2}{15})^3=(\frac{28}{15})^3$$

and $\quad y^3= \quad 2^3=8$

which are three cubes fulfilling the required conditions of the question.

Otherwise. Let x^3, y^3, b^3, be the three required cubes, and a the given number; then it is obvious that the question only requires that $x^3+y^3+b^3-3a=m^2$.

And here one of the cubes b^3 may be taken at pleasure; also a being given, we may consider $b^3-3a=\pm c$, a given number; whence we have to find $x^3+y^3\pm c=m^2$:

Let $\quad x=d+z$, and $y=f-z$;

Then $\quad x^3=d^3+3d^2z+3dz^2+z^3$

$$y^3=f^3-3f^2z+3fz^2-z^3,$$

By addition, $d^3+f^3+3(d^2-f^2)z+3(d+f)z^2\pm c=m^2$

$$3(d+f)z^2+3(d^2-f^2)z+d^3+f^3\pm c=m^2;$$

Which latter expression may always be made a square, provided $3(d+f)=$ a square.

Assume $d=2$, and $f=1$, then $3(d+f)=9$, and the above becomes $9z^2+9z+9\pm c=m$,

Let this $=(3z+r)^2=9z^2+6rz+r^2$,

Then $9z+9\pm c=6rz+r^2$, and

$$z=\frac{9\pm c-r^2}{6r-9},$$

which is similar to the preceding expression.

SUMMATION AND INTERPOLATION

OF

INFINITE SERIES.

PROBLEM I.

Any series being given, to find the several orders of differences.

Ex. 3.	Here, 1, 3, 6, 10, 15, 21,	given series.
	2, 3, 4, 5, 6,	first diff.
	1, 1, 1, 1.	
Ex. 4.	Here 1, 6, 20, 50, 105, 196, &c.	given series.
	5, 14, 30, 55, 91,	1st diff.
	2, 16, 25, 36,	2d diff.
	7, 9, 11,	3d diff.
	2, 2, &c.	4th diff.
Ex. 5.	Here $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$, &c.	the given series.
	$\frac{-1}{4}, \frac{-1}{8}, \frac{-1}{16}, \frac{-1}{32}$, &c.	1st diff.
	$\frac{1}{8}, \frac{1}{16}, \frac{1}{32}$, &c.	2d diff.
	$\frac{-1}{16}, \frac{-1}{32}$, &c.	3d diff.
	$\frac{1}{32}$, &c.	4th diff.
	&c.	&c.

PROBLEM II.

Any series, a, b, c, d, e, &c. being given, to find the first term of the nth *order of differences.*

Ex. 3. Here a, b, c, d, e, &c. are respectively 1, 3, 9, 27, 81, &c. also $n=8$.

Whence $a-nb+\frac{n(n-1)}{1.2}c-\frac{n(n-1)(n-2)}{1.2.3}d+$&c.

$=1-8b+28c-56d+70e-56f+28g-8h+i$
$=1-24+252-1512+5670-13608+$
$20412-17496+6561=32896-32640=256$.

Ex. 4. Here $a=1$, $b=\frac{1}{2}$, $c=\frac{1}{4}$, $d=\frac{1}{8}$, &c.

Also $n=5$; whence

$-a+nb-\frac{n(n-1)}{1.2}c+\frac{n(n-1)(n-2)}{1.2.3}d-$&c.$=$

$-1+5b-10c+10d-5e+f=$

$-1+\frac{5}{2}-\frac{10}{4}+\frac{10}{8}-\frac{5}{16}+\frac{1}{32}=$

$-\frac{32}{32}+\frac{31}{32}=\frac{-1}{32}$ Answer.

PROBLEM III.

To find the nth *term of the series a, b, c, d, e, &c. when the differences of any order become at least equal to each other.*

Ex. 3. Here 1, 4, 9, 16, 25, 36, &c.

3, 5, 7, 9, 11,	1st diff.
2, 2, 2, 2,	2d diff.
0, 0, 0,	3d diff.

Therefore $d'=3$, $d''=2$, $d'''=0$, and $n=15$;

Whence

$$a+\frac{n-1}{1}d'+\frac{(n-1)(n-2)}{1.2}d''+\frac{(n-1)(n-2)(n-3)}{1.2.3}$$

$$d'''+\&c.$$

$$=1+14d'+91d''=1+42+182=225,\ \textbf{Answer.}$$

Ex. 4. Here 1, 8, 27, 64, 125, 216, &c.

7, 19, 37, 61, 91,	&c.	1st diff.
12, 18, 24, 30,	&c.	2d diff.
6, 6, 6,	&c.	3d diff.
0, 0,	&c.	4th diff.

Whence, since $d'=7$, $d''=12$, $d'''=6$, and $n=20$,

we have

$$a+\frac{n-1}{1}d'+\frac{(n-1)(n-2)}{1.2}d''+\frac{(n-1)(n-2)(n-3)}{1.2.3}$$

$$d'''+\&c.$$

$$=1+19d'+171d''+969d'''$$

$$=1+133+2052+5814=8000\ \text{Ans.}$$

Ex. 5. It will here be sufficient if we take merely the denominators of the proposed fractions, viz.

1, 3, 6, 10, 15, &c.

2, 3, 4, 5,	&c.	1st diff.
1, 1, 1,	&c.	2d diff.
0, 0,	&c.	3d diff.

Whence, $d'=2$, and $d''=1$, also $n=30$,

Therefore $a+\frac{n-1}{1}d'+\frac{(n-1)(n-2)}{1.2}d''=$

$1+29d'+406d''=$

$1+58\ +406\ =465$;

Consequently $\frac{1}{465}$ is the 30th term sought.

PROBLEM IV.

To find the sum of n terms of the series a, b, c, d, e, &c. when the differences of any order become at last equal to each other.

Ex. 4. Here 2, 6, 12, 20, 30, &c.
4, 6, 8, 10, &c. 1st diff.
2, 2, 2, &c. 2d diff.
0, 0, &c. 3d diff.

Therefore $a=2$, $d'=4$, $d''=2$;

Consequently $na+\frac{n(n-1)}{1.2}d'+\frac{n(n-1)(n-2)}{1.2.3}d''=2n+\frac{n(n-1)}{1.2}\times4+\frac{n(n-1)(n-2)}{1.2.3}\times2=2n^2+\frac{n(n^2-3n+2)}{3}=$

$\frac{n(n+1)(n+2)}{3}$ Answer.

Ex. 5. Here 1, 3, 6, 10, 15, &c. given series.
2, 3, 4, 5, &c. 1st diff.
1, 1, 1, &c. 2d diff.
0, 0, &c. 3d diff.

Consequently $a=1$, $d'=2$, $d''=1$,

Whence we have

$$na+\frac{n(n-1)}{1.2}d'+\frac{n(n-1)(n-2)}{1.2.3}d''=$$

$$n+\frac{n(n-1)}{1.2}\times2+\frac{n(n-1)(n-2)}{1.2.3}=$$

$$n^2+\frac{n(n^2-3n+2)}{6}=\frac{n(n+1)(n+2)}{1.2.3}$$ Answer.

Ex. 6. Here 1, 4, 10, 20, 35, &c.
3, 6, 10, 15, &c. 1st diff.
3, 4, 5, &c. 2d diff.
1, 1, &c. 3d diff.

Whence $a=1$, $d'=3$, $d''=3$, $d'''=1$; and therefore

$$na+\frac{n(n-1)}{1.2}d'+\frac{n(n-1)(n-2)}{1.2.3}d''+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}d'''$$

$$=n+\frac{3n(n-1)}{2}+\frac{3n(n-1)(n-2)}{1.2.3}+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}=\,\cdots$$

$$\frac{3n^2-n}{2}+\frac{n(n^2-3n+2)}{2}+\frac{n(n^3-6n^2+11n-6)}{2.3.4}=\,\cdots\cdots$$

$\dfrac{n(n^3+6n^2+11n+6)}{2.3.4}=\dfrac{n}{1}\times\dfrac{n+1}{2}\times\dfrac{n+2}{3}\times\dfrac{n+3}{4}$, Answer as required.

Ex. 7. Instead of n terms of the proposed series, let us find the sum of $n+1$ terms of the series 0, 1, 16, 81, &c. which will reduce the magnitude of the successive differences. Then we shall have

0,	1,	16,	81,	256,	&c.
	1,	15,	65,	175,	&c. 1st diff.
		14,	50,	110,	&c. 2d diff.
			36,	60,	&c. 3d diff.
				24,	&c. 4th diff.

Whence $a=0$, $d'=1$, $d''=14$, $d'''=36$, $d^{iv}=24$, and the number of terms $=n+1$.

Whence, by substituting $n+1$ for n, and the above values of a, d', d'', d''', &c. in the general formula,

$$na+\frac{n(n-1)}{1.2}d'+\frac{n(n-1)(n-2)}{1.2.3}d''\ \&c.$$

we shall have

$$\frac{n^2+n}{2}+\frac{n^3-n}{2.3}\times 14+\frac{n^4-2n^3-n^2+2n}{2.3.4}\times 36+$$

$$\frac{n^5-5n^4+5n^3+5n^2-6n}{1.2.3.4.5}\times 24=$$

$\dfrac{n^5}{5}+\dfrac{n^4}{2}+\dfrac{n^3}{3}-\dfrac{n}{30}$, the answer.

Ex. 8. Proceeding here exactly as in the preceding example, we have to find the sum of $n+1$ terms of the series, 0, 1^5, 2^5, 3^5, &c. that is, of

0,	1,	32,	243,	1024,	3125,	&c.
1,	31,	211,	781,	2101,		&c. 1st diff.
30,	180,	570,	1320,			&c. 2d diff.
150,	390,	750,				&c. 3d diff.
240,	360,					&c. 4th diff.
120,						&c. 5th diff.

Whence

$a=0$, $d'=1$, $d''=30$, $d'''=150$, $d^{iv}=240$, and $d^{v}=120$;

And by substituting these values in the general formula, and $n+1$ for n,

we shall have

$$\frac{n^6}{6}+\frac{n^5}{2}+\frac{5n^4}{12}-\frac{n^2}{12}=\text{ the sum required.}$$

PROBLEM V.

The series a, b, c, d, e, &c. being given, whose terms are an unit's distance from each other, to find any intermediate term by interpolation.

Ex. 2. In order to save the trouble of reduction, take the logarithms of the numbers; in which case we shall have

Terms.	*Logarithms.*	d',	d'',	d''',	d^{iv}.
a, $\frac{1}{50}$	$=-1.6989700$				
		86002			
b, $\frac{1}{51}$	$=-1.7075702$		1671		
		84331		66	
c, $\frac{1}{52}$	$=-1.7160033$		1605		8
		82726		58	
d, $\frac{1}{53}$	$=-1.7242759$		1547		
		81179			
e, $\frac{1}{54}$	$=-1.7323938$.				

Here the first terms of the differences are $d'=86002$, $d''=1671$, $d'''=66$, $d^{iv}=8$; and the distance of y, the term to be interpolated, being $2\frac{1}{2}$, we have

$$y=a+xd'+\frac{x(x-1)}{1.2}d''+\frac{x(x-1)\,(x-2)}{1.2.3}d'''+\&c.$$

$$=a+\frac{5}{2}d'+\frac{15}{8}d''+\frac{5}{16}d'''-\frac{5}{128}d^{iv}$$

$$=-1.6989700-215005+3133-21-1$$

$$=-1.7201594=\text{ log. of }\frac{1}{52.5}=\frac{2}{105}\text{ Answ.}$$

Ex. 3. Given the natural tangents of 88° 54′, 88° 55′, 88° 56′, 88° 57′, 88° 58′, 88° 59′ and 89°, to find that of 88° 58′ 18″.

Tangents.	d'	d''	d'''	d^{iv}	d^{v}	d^{vi}
$a=52.080673$						
	801436					
$b=52.882109$		25042				
	826478		1193			
$c=53.708587$		26235		76		
	852713		1296		7	
$d=54.561300$		27504		83		1
	880217		1352		8	
$e=55.441517$		28856		91		
	909073		1443			
$f=56.350590$		30299				
	939372					
$g=57.289962$						

Again (88° 58′ 18″)—(88° 54′)=4′ 18″=4′ .3,

Whence $x=4'.3$; and consequently

$$y=a+4.3d'+\frac{4.3\times3.3}{2}d''+\frac{4.3\times3.3\times2.3}{6}d'''+$$

$$\frac{4.3\times3.3\times2.3\times1.3}{24}d^{iv}+\frac{4.3\times3.3\times\&c.\ .3\times-.7d^{v}}{120}$$

And consequently, by collecting the terms,

$$\begin{array}{r} 52.080673 \\ 3.446175 \\ 0.177673 \\ 6489 \\ 134 \\ \hline 55.711144 \end{array} = \text{tangent } 88^\circ\ 58'\ 18''$$

The answer required.

PROBLEM VI.

Having given a series of equidistant terms, a, b, c, d, e, &c. whose first differences are small, to find any intermediate term by interpolation.

Ex. 2. Given the cube roots of 45, 46, 47, 48, and 49, to find the cube root of 50.

Here the number of terms are 5; and against 5 in the tablet we have $a-5b+10c-10d+5e-f=0$, or

$$f=a-5b+10c-10d+5e.$$

Num.	Roots.
$\sqrt[3]{45}$	$=3.556893=a$
$\sqrt[3]{46}$	$=3.584048=b$
$\sqrt[3]{47}$	$=3.608826=c$
$\sqrt[3]{48}$	$=3.634241=d$
$\sqrt[3]{49}$	$=3.659306=e$

Hence $a=\ 3.556893$ $\quad$ $5b=17.915240$

$10c=36.088260$ $\quad$ $10d=36.342410$

$5e=18.296530$

Sum $+57.941683$ $\quad$ 54.257650

Sum -54.257650

$3.684033=$ the cube root of 50

Ex. 3. Here, following the same process as before, we have

$a=\log.\ 50=1.6989700$
$b=\log.\ 51=1.7075702$
$c=\log.\ 52=1.7160033$
$d=\log.\ 53$ *required.*
$e=\log.\ 54=1.7323938$
$f=\log.\ 55=1.7403627$
$g=\log.\ 56=1.7481880$

Also the number of terms being 6, we have from the tablet

$$a-6b+15c-20d+15e-6f+g=0,$$

$$\text{or } d=\frac{a-6b+15c+15e-6f+g}{20};$$

Whence, collecting the terms, we shall obtain

$$\begin{aligned} a &= 1.6989700 \\ 15c &= 25.7400495 \\ 15e &= 25.9859070 \\ g &= 1.7481880 \\ \hline \text{Sum } &+55.1731145 \\ -6(b+f) &= 20.6875974 \\ \hline \text{Difference } &= 34.4855171 \end{aligned}$$

Therefore $34.4855171 \div 20 = 1.72427585 =$ the log. of 53, as required.

EXAMPLES FOR PRACTICE.

Ex. 1. Here, by the differential method,

$$\begin{array}{llll} 2, & 5, & 8, & 11, \ \&c. \text{ given series} \\ & 3, & 3, & 3, \ \&c. \text{ 1st diff.} \end{array}$$

Now $n=100$, $a=2$, and $d'=3$; therefore

$$s=na+\frac{n(n-1)}{1.2}d'=2\times 100+\frac{100\times 99}{2}\times 3,$$

or $s=200+14850=15050$, the sum required.

Ex. 2. Here, by the differential formula,

$$\begin{array}{lllll} 1, & 4, & 9, & 16, & \&c. \\ & 3, & 5, & 7, & \&c. \text{ 1st diff.} \\ & & 2, & 2, & \&c. \text{ 2d diff.} \end{array}$$

Whence $a=1$, $d'=3$, $d''=2$, and $n=50$;

Therefore $s=na+\frac{n(n-1)}{1.2}d'+\frac{n(n-1)(n-2)}{1.2.3}d''$,

or $s=50+1225d'+19600d''$,

or $s=50+3675+39200=42925$.

Ex. 3. Let $\frac{z}{(1-x)^3}=s=1+3x+6x^2+10x^3+$ &c.

Then $z=(1-x)^3\times(1+3x+6x^2+10x^3+$ &c.)
Which, by actual multiplication, is $=1$; therefore

$$z=1, \text{ and } s=\frac{1}{(1-x)^3}, \text{ as required.}$$

Ex. 4. Let $\frac{z}{(1-x)^4}=s=1+4x+6x^2+20x^3+35x^4+$ &c.

Then $z=(1-x)^4\times(1+4x+6x^2+20x^3+35x^4+$ &c.) $=1$, as appears by the actual operation;

Whence $z=1$, and $s=\frac{1}{(1-x)^4}$ the sum required.

Ex. 5. Let $z=\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+$, &c.

Then $z-1=\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+$ &c.

And by subtraction,

$$1=\frac{2}{1.3}+\frac{2}{3.5}+\frac{2}{5.7}+\frac{2}{7.9}+\frac{2}{9.11} \text{ \&c.}$$

$$\text{or } =2\left(\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\frac{1}{9.11} \text{ \&c.}\right)$$

Consequently by division,

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9} \text{ \&c.}=\frac{1}{2},$$

which is the sum required.

Ex. 6. Let the given or proposed series

$$1.2+3.4+5.6+7.8+\text{\&c.}$$

be put under the form

$$(1^2+1)+(3^2+3)+(5^2+5)+\text{\&c.}$$

$$\text{or } \begin{cases} 1^2+3^2+5^2+7^2+\text{\&c.} \\ 1\ +3\ +5\ +7\ +\text{\&c.} \end{cases}$$

Then, by differencing in the first, we shall have
$d'=8$, $d''=8$, $a=1$, and $n=40$;

Whence $na+\frac{n(n-1)}{2}d'+\frac{n(n-1)(n-2)}{2.3}d''=s$;

And consequently

$$4+780d'+9880d''=$$
$$4+6240+79040=85284$$

Also $1+3+5+7+\&c. =1600$:

Therefore, by addition, the sum required $=86884$.

Ex. 7. Here the given, or proposed series

$$\frac{2x-1}{2x}+\frac{2x-3}{2x}+\frac{2x-5}{2x}+\frac{2x-7}{2x}+\&c.$$

may be separated into the two

$$\frac{2x}{2x}+\frac{2x}{2x}+\frac{2x}{2x}+\frac{2x}{2x}+\&c.$$

$$-\frac{1}{2x}(1+3+5+7+9+\&c.)$$

The sum of n terms of the former of which $=n$, and the same number of the latter $=\frac{n^2}{2x}$;

Whence $n-\frac{n^2}{2x}=\frac{2nx-n^2}{2x}=n(\frac{2x-n}{2x})$, the sum required.

Ex. 8. Let $s=\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}+\&c.$

Then $s-\frac{1}{6}=\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}$, by transposition;

Whence $\frac{1}{6}=\frac{1}{1.2.3.4}+\frac{3}{2.3.4.5}+\frac{3}{3.4.5.6}+$, &c.

by subtraction;

Consequently $\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}+\frac{1}{4.5.6.7}$
$+$ &c. $=$

$\frac{1}{6}\div 3=\frac{1}{18}$, the sum required.

Ex. 9. Here the denominators of the fractions are figurate numbers of the third order; and if we divide the proposed series by 6, it will become

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}+\text{ \&c.}$$

Assume $z=\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+$ &c.

Then $z-\frac{1}{2}=\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\frac{1}{5.6}+$ &c.

And by subtraction,

$$\frac{1}{2}=\frac{2}{1.2.3}+\frac{2}{2.3.4}+\frac{2}{3.4.5}+\frac{2}{4.5.6}+\text{ \&c.}$$

Whence by division,

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}+\text{ \&c.}=\frac{1}{4}$$

Consequently our proposed series, which is six times this, viz.

$$\frac{1}{1}+\frac{1}{4}+\frac{1}{10}+\frac{1}{20}+\frac{1}{35}+\text{ \&c.}=\frac{6}{4}=1\frac{1}{2}$$

Ex. 10. Let $\frac{z}{(1-x)^4}=1+2^3x+3^3x^2+4^3x^3+5^3x^4+\text{\&c.})$

Then $z=(1-x)^4\times(1+2^3x+3^3x^2+4^3x^3+5^3x^4+\text{\&c.})$

$$\text{Or}\begin{cases}(1+2^3x+3^3x^2+4^3x^3+\text{\&c.})\times\\(1-4\ x+6\ x^2-4\ x^3+x^4)\end{cases}$$

$$z=\begin{cases}1+8x+27x^2+\ 64x^3+\text{ \&c.}\\ \quad -4x-32x^2-103x^3-\text{ \&c.}\\ \qquad\quad +\ 6x^2+\ 48x^3+\text{ \&c.}\\ \qquad\qquad\quad -\ \ 4x^3-\text{ \&c.}\end{cases}$$

$$=1+4x+x^2+0+0+\text{ \&c.}$$

Whence $\frac{1+4x+x^2}{(1-x)^4}=$ the sum required.

Ex. 11. Let $x=\frac{1}{r}$, and $s=\frac{z}{(r-1)^2}$;

Then $\frac{z}{(r-1)^2}=\frac{1}{r}+\frac{2}{r^2}+\frac{3}{r^3}+\frac{4}{r^4}+\frac{5}{r^5}$&c.

And $z=(r-1)^2\times(\frac{1}{r}+\frac{2}{r^2}+\frac{3}{r^3}+\frac{4}{r^4}+$&c.)

Which, by the actual operation, becomes $=r$; hence $z=r$; and the sum of the series continued to infinity is

$$\frac{r}{(r-1)^2}$$

Now, for the other part, the terms after the nth are,

$$\frac{n+1}{r^{n+1}}+\frac{n+2}{r^{n+2}}+\frac{n+3}{r^{n+3}}+\text{\&c.}$$

$$\text{or } \frac{1}{r^n}\left(\frac{n+1}{r}+\frac{n+2}{r^2}+\frac{n+3}{r^3}+\text{\&c.}\right)=$$

$$\frac{1}{r^n}\left\{\frac{(1-\frac{1}{r})r+\frac{n}{r}}{(r-\frac{1}{r})^2}-n\right\}=$$

$$\frac{1}{r^n}\left\{\frac{n}{r-1}+\frac{r}{(r-1)^2}\right\}$$

Whence, by subtracting this from the whole sum before found, we have

$\frac{r}{(r-1)^2}-\frac{1}{r^n}(\frac{n}{r-1}+\frac{r}{(r-1)^2})=\frac{(1-p)r}{(r^2-1)^2}-\frac{pn}{r-1}=$ the sum of n terms, as required; where $p=\frac{1}{r^n}$.

Ex. 12. Let $z=\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+$&c.

Then $z-\frac{1}{2}-\frac{1}{4}=\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\frac{1}{12}+\frac{1}{14}+$&c.

And therefore by subtraction

$$\frac{3}{4}=\frac{4}{2.6}+\frac{4}{4.8}+\frac{4}{6.10}+\frac{4}{8.12}+\&c.$$

Whence $z=\frac{3}{16}=\frac{1}{2.6}+\frac{1}{4.8}+\frac{1}{6.10}+\frac{1}{8.12}+\&c.$

The infinite sum required.

Again, let $z=\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\ldots\frac{1}{2n}$

Then $z-\frac{3}{4}=\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\frac{1}{12}\ldots\frac{1}{2n}$,

And $z-\frac{3}{4}+\frac{1}{2n+2}+\frac{1}{2n+4}=\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\frac{1}{12}+$

$$\frac{1}{14}+\&c.\frac{1}{2n+4};$$

Whence by subtraction,

$$\frac{3}{4}-\frac{1}{2n+2}-\frac{1}{2n+4}=\frac{4}{2.6}+\frac{4}{4.8}+\frac{4}{6.10}+\frac{4}{8.12}$$

$+\&c.$ to n terms;

Consequently by division,

$$\frac{1}{2.6}+\frac{1}{4.8}+\frac{1}{6.10}+\frac{1}{8.12}+\&c. \text{ to } n \text{ terms},$$

$$=\frac{3}{16}-\frac{1}{8(n+1)}-\frac{1}{8(n+2)}=\frac{5n+3n^2}{16n^2+48n+32}=z,$$

the sum of n terms as required.

Ex. 13. Let $z=\frac{1}{3}+\frac{1}{6}+\frac{1}{9}+\frac{1}{12}+\&c.$

Then $z-\frac{1}{3}=\frac{1}{6}+\frac{1}{9}+\frac{1}{12}+\frac{1}{15}+\&c.$

And by subtraction,

$$\frac{1}{3}=\frac{3}{3.6}+\frac{3}{6.9}+\frac{3}{9.12}+\frac{3}{12.15}+\&c.$$

$$\text{Or } \frac{1}{3}=\frac{1}{2.3}+\frac{1}{3.6}+\frac{1}{4.9}+\frac{1}{5.12}+, \&c.$$

$$\text{Therefore, } \frac{1}{3}\times\frac{1}{4}=\frac{1}{12}=\frac{1}{3.8}+\frac{1}{6.12}+\frac{1}{9.16}+\frac{1}{12.20}+, \&c.$$

$$\text{Again, let } z=\frac{1}{3}+\frac{1}{6}+\frac{1}{9}+\frac{1}{12}+, \&c. \ldots \frac{1}{3n}$$

$$\text{Then, } z-\frac{1}{3}=\frac{1}{6}+\frac{1}{9}+\frac{1}{12}+\frac{1}{15}+, \&c. \ldots \frac{1}{3n}$$

$$\text{And } z-\frac{1}{3}+\frac{1}{3n+3}=\frac{1}{6}+\frac{1}{9}+\frac{1}{12}+\frac{1}{15}+\frac{1}{3n+3}$$

Whence, by subtraction,

$$\frac{1}{3}-\frac{1}{3(n+1)}=\frac{1}{3.2}+\frac{1}{6.3}+\frac{1}{9.4}+\frac{1}{12.5}+, \&c.\ n \text{ terms.}$$

And, dividing by 4,

$$\frac{1}{3.8}+\frac{1}{6.12}+\frac{1}{9.16}+\frac{1}{12.20}+, \&c. \text{ to } n \text{ terms.}$$

$$=\frac{1}{12}-\frac{1}{12(n+1)}=\frac{n}{12(n+1)}=s,$$

the sum of n terms as required.

$$\text{Ex. 14. Let } z=\frac{1}{2}+\frac{1}{7}+\frac{1}{12}+\frac{1}{17}+, \&c.$$

$$\text{Then, } z-\frac{1}{2}=\frac{1}{7}+\frac{1}{12}+\frac{1}{17}+\frac{1}{22}+, \&c.$$

And, by subtraction,

$$\frac{1}{2}=\frac{5}{2.7}+\frac{5}{7.12}+\frac{5}{12.17}+\frac{5}{17.22}+, \&c.$$

Whence, multiplying by $\frac{6}{5}$, we have

$$\frac{6}{2.7}+\frac{6}{7.12}+\frac{6}{12.17}+\frac{6}{17.22}+, \&c. \text{ ad. inf.} =\frac{3}{5}$$

Now, the general term of the series is

$$\frac{6}{(5n-3)(5n+2)}$$

And therefore, to find the sum of all the terms beyond this, we need only assume

$$z=\frac{1}{5n+2}+\frac{1}{5n+7}+\frac{1}{5n+12}+\&c.$$

$$\text{Then } z-\frac{1}{5n+2}=\frac{1}{5n+7}+\frac{1}{5n+12}+\frac{1}{5n+17}+\&c.$$

And, by subtraction,

$$\frac{1}{5n+2}=\frac{5}{(5n+2)(5n+7)}+\frac{5}{(5n+7)(5n+12)}+\&c.$$

Or, multiplying by $\frac{6}{5}$, we have

$$\frac{6}{(5n+2)(5n+7)}+\frac{6}{(5n+7)(5n+12)}+\&c.=\frac{6}{5(5n+2)};$$

Which latter expression is the sum of all the terms after the nth; consequently

$$\frac{3}{5}-\frac{6}{5(5n+2)}=\frac{3n}{5n+2}, \text{ the sum of } n \text{ terms.}$$

Ex. 15. This series may be put under the form

$$\frac{1}{6}\left(\frac{5}{1.2.3.4}+\frac{9}{3.4.5.6}+\frac{13}{5.6.7.8}+\&c.\right)$$

Let us therefore assume

$$z=\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+\frac{1}{7.8}+\&c.$$

$$\text{Then, } z-\frac{1}{2}=\frac{1}{3.4}+\frac{1}{5.6}+\frac{1}{7.8}+\frac{1}{9.10}+\&c.$$

$$\text{By subtraction } \frac{10}{1.2.3.4}+\frac{18}{3.4.5.6}+\frac{26}{5.6.7.8}\&c.=\frac{1}{2};$$

Or, by dividing by 2,

$$\frac{5}{1.2.3.4}+\frac{9}{3.4.5.6}+\frac{13}{5.6.7.8}+\&c.=\frac{1}{4};$$

and, consequently,

$$\frac{1}{6}\left(\frac{5}{1.2.3.4}+\frac{9}{3.4.5.6}+\frac{13}{5.6.7.8}+, \&c.\right)=\frac{1}{24},$$

$$\text{or } \frac{1}{3.6}-\frac{1}{6.8}+\frac{1}{9.10}-\frac{1}{12.12}+, \&c. =\frac{1}{24} \text{ inf. sum.}$$

Again, if the above assumed series be carried beyond the nth term, it becomes

$$\frac{1}{(2n+2)(2n+1)}+\frac{1}{(2n+4)(2n+3)}+, \&c.$$

Whence, as above, $\frac{1}{12(2n+2)(2n+1)}=$ inf. sum.

after n terms; consequently,

$$\frac{1}{24}-\frac{1}{12(2n+2)(2n+1)}=\frac{n}{2(3+6n)}-\frac{n}{4(6+6n)}$$

$$= \text{sum of } n \text{ terms.}$$

Ex. 16. Assume

$$z=\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{9}+\frac{1}{11}-, \&c.$$

$$\text{Then, } z-\frac{1}{3}=\frac{-1}{5}+\frac{1}{7}-\frac{1}{9}+\frac{1}{11}-\frac{1}{13}+, \&c.$$

$$\text{Subtracting, } \frac{1}{3}=\frac{8}{3.5}-\frac{12}{5.7}+\frac{16}{7.9}-\frac{20}{9.11}+\frac{24}{11.13}+, \&c.$$

Dividing by 4,

$$\frac{2}{3.5}-\frac{3}{5.7}+\frac{4}{7.9}-\frac{5}{9.11}+, \&c. =\frac{1}{12} \text{ inf. sum.}$$

Again, if the above assumed series be continued beyond the nth term, it will be

$$z'=\pm\frac{1}{3+2n}\mp\frac{1}{5+2n}\pm\frac{1}{7+2n}\mp\frac{1}{9+2n}\pm, \&c.$$

And therefore, as before, $\pm \frac{1}{4(3+2n)}$ will be the infinite sum of the terms after the nth:

Consequently $\frac{1}{12} \pm \frac{1}{4(3+2n)} =$ sum of n terms required; the ambiguous sign being $+$ when n is odd, and $-$ when n is even.

Ex. 17. Assume $z = \frac{3}{1.2} + \frac{4}{2.3} + \frac{5}{3.4} + \frac{6}{4.5} +$ &c.

Then $z - \frac{3}{2} = \frac{4}{2.3} + \frac{5}{3.4} + \frac{6}{4.5} + \frac{7}{5.6} +$ &c.

Subtracting $\frac{3}{2} = \frac{5}{1.2.3} + \frac{6}{2.3.4} + \frac{7}{3.4.5} + \frac{8}{4.5.6} +$ &c. the infinite sum.

Again, if the above assumed series be carried beyond the nth term, it will be

$$z' = \frac{n+3}{(n+1)(n+2)} + \frac{n+4}{(n+2)(n+3)} + \frac{n+5}{(n+3)(n+4)}$$

Whence, as before, the first term $\frac{n+3}{(n+1)(n+2)}$ will be the infinite sum of the terms past the nth.

Therefore $\frac{3}{2} - \frac{n+3}{(n+1)(n+2)}$;

or, $\frac{3}{2} - \frac{2}{n+1} + \frac{1}{n+2} =$ the sum of n terms.

LOGARITHMS.

MULTIPLICATION BY LOGARITHMS.

Ex. 7.	Log. of	23.14	=	1.3643634
	log. of	5.062	=	0.7043221
	Prod.	117.1347		2.0686855
Ex. 8.	Log. of	4.0763	=	0.6102661
	Log. of	9.8432	=	0.9931363
	Prod.	40.12383		1.6034024
Ex. 9.	Log. of	498.256	=	2.6974525
	log. of	41.2467	=	1.6153892
	Prod.	20551.41		4.3128417
Ex. 10.	Log. of	4.26747	=	0.6049541
	log. of	.012345	=	—2.0914911
	Prod.	.04971016		—2.6964452
Ex. 11.	Log. of	3.12567	=	0.4949431
	log. of	.02868	=	—2.4575791
	log. of	.12379	=	—1.0926856
	Prod.	.01109705		—2.0452078
Ex. 12.	Log. of	2876.9	=	3.4589248
		.10674	=	—1.0283272
		.098762	=	—2.9945899
		.0031598	=	—3.4996596
	Prod.	.0958299		—2.9815015

DIVISION BY LOGARITHMS.

Ex. 7.	Log. of 125	=	2.0969100
	log. of 1728	=	3.2375437
	Quot. .0723379		—2.8593663
Ex. 8.	Log. of 1728.95	=	3.2377825
	log. of 1.10678	=	0.0440613
	Quot. 1562.144		3.1937212
Ex. 9.	Log. of 10.23674	=	1.0101617
	log. of 4.96523	=	0.6959393
	Quot. 2.061685		0.3142224
Ex. 10.	Log. of 19956.7	=	4.3000888
	log. of .048235	=	—2.6833623
	Quot. 413719		5.6167265
Ex. 11.	Log. of .067859	=	—2.8316075
	log. of 1234.59	=	3.0915228
	Quot. .0000549648		—5.7400847

RULE OF THREE BY LOGARITHMS.

Ex. 5.	Comp. log. of 12.678	=	8.8969493
	log. of 14.065	=	1.1481397
	log. of 100.979	=	2.0042311
	Ans. 112.0263		2.0493201
Ex. 6.	Comp. log. of 1.9864	=	9.7019333
	log. of .4678	=	—1.6700602
	log. of 50.4567	=	1.7029274
	Ans. 11.88262		1.0749209

Ex. 7. Comp. log. of .09658= 11.0151127
log. of .24958=— 1.3972098
log. of .008967=— 3.9526472

Ans. .02317234 — 2.3649697

Ex. 8. The first part of this example is misprinted; it should have been a third proportional, not a mean proportional.

Comp. log. of .498621=—10.3022294
log. of 2.9587= 0.4711009
log. of 2.9587= 0.4711009

3d prop. 17.55623 1.2444312

Whence, 17.55623, the answer required.

Comp. log. of 12.796=8.8929258
log. of 3.24718=0.5115063
log. of 3.24718=0.5115063

3d prop. .8240212 —1.9159384

INVOLUTION BY LOGARITHMS.

Ex. 5. Log. of 6.05987=0.7824633
2

Ans. 36.72203 1.5649266

Ex. 6. Log. of .176546=—1.2468579
3

Ans. .005502674 —3.7405737

Ex. 7. Log. .076543=—2.8839055
4

Ans. .00003432594 —5.5356220

Ex. 8.	Log. of 2.97643=0.4736957		
			5
		Ans. 233.6031	2.3684785
Ex. 9.	Log. of 21.0576=1.3234089		
			6
		Ans. 87187340	7.9404534
Ex. 10.	Log. of 1.09684=0.0401432		
			7
		Ans. 1.909864	0.2810024

EVOLUTION BY LOGARITHMS.

Ex. 7.	Log. of 365.5674		2)2.5629674
		Ans. 19.11981	1.2814837
Ex. 8.	Log. of 2.987635		3)0.4753276
		Ans. 1.440265	0.1584425
Ex. 9.	Log. of .967845		4)—1.9858059
		Ans. .9918824	—1.9964515
Ex. 10.	Log. of .098674		7)—2.9942027
		Ans. .7183146	—1.8563147

Ex. 11. Log. of 21 = 1.3222193
log. of 373 = 2.5717088

—2.7505105
Multiplying by 2

Dividing by 3)—3.5010210

Ans. .146895 =—1.1670070

Ex. 12. Log. of 112 = 2.0492180
log. of 1727 = 3.2372923

—2.8119257

Which, being multiplied by 3, and then divided by 5, gives log. —1.2871554, Ans. .1937115.

MISCELLANEOUS EXAMPLES.

Ex. 1. Log. of 2 = 0.3010300
log. of 123 = 2.0809051

2)—2.2111249

Ans. .1275153 —1.1055624

Ex. 2. 0—log. 3.14159 3)—1.5028505

.6827842 —1.8342835

Ex. 3. Log. .00563 = —3.7505084
.07=$\frac{7}{100}$, theref. 7

100)—16.2535588

Ans. .6958321 —1.8425355

The student will observe, that 84 is borrowed, in this example, to make the 16 up to 100, according to the rule.

Ex. 4. $\dfrac{(\frac{2}{3})^{\frac{1}{2}}\times(\frac{3}{4})^{\frac{1}{3}}}{17\frac{1}{3}}=(\frac{8}{27}\times\frac{9}{16})^{\frac{1}{6}}\div\frac{52}{3}=(\frac{1}{6})^{\frac{1}{6}}\times\frac{3}{52}$

Hence, 0—log. 6	6)—1.2218487
	—1.8703081
Log. of $\frac{3}{52}$	—2.7611180
Ans. .04279825	—2.6314261

Ex. 5. Log. of $\frac{1}{7}$ =	—1.1549020
$\frac{1}{2}$ log. of $\frac{5}{8}$ =	—1.8979400
log. of .012 =	—2.0791812
$\frac{1}{3}$ log. $\frac{7}{11}$ =	—1.9345684
Ans. .001165713	—3.0665916

Ex. 6. Log. of $\frac{1}{9}$ =	—1.0457574
$\frac{1}{2}$ log. of $\frac{11}{21}$ =	—1.8595867
log. .03 =	—2.4771213
$\frac{1}{3}$ log. .15$\frac{1}{5}$ =	0.3939479
	—3.7764133
log. of $\frac{22}{3}$ =	0.8653014
$\frac{1}{3}$ log. 12$\frac{1}{5}$ =	0.3621199
log. .19 =	—1.2787536
$\frac{1}{4}$ log. 17$\frac{1}{8}$ =	0.3084076
	0.8145825
From	—3.7764133
Subtract	0.8145825
Ans. .0009158636	—4.9618308

Ex. 7. Log. of $\frac{1}{6}$ $=-1.2208187$
$\frac{1}{2}$ log. 19 $=\ 0.6393768$

Nat. numb. .7247622—1.8601955

Log. $\frac{4}{7}$ $=-1.7569620$
$\frac{1}{3}$ log. $\frac{106}{3}$ $=\ 0.5160615$

Nat. numb. 1.875096—0.2730235

Log. of 2.5998582 $=\ 0.4149496$
log. of $\frac{127}{4}$ $=\ 1.5017437$

1.9166933

In the same manner we find
log. $(14\frac{7}{9}-\frac{1}{11}\sqrt[5]{28\frac{2}{3}})=0.2230796$

Ans. 49.38712 1.6936137

MISCELLANEOUS QUESTIONS.

Ex. 1. Let x be the number of minutes after 8, or the number the minute hand is before it overtakes the hour hand, after the former is at 12, and the latter at 8.

Then, $\frac{x}{12}$ will be the number of minutes that the hour hand has advanced in the same time. And, by the question,

$$40+\frac{1}{12}x=x,\ \text{or}\ \frac{11}{12}x=40,$$

$$\text{or}\ x=\frac{40\times 12}{11}=43\frac{7}{11};$$

viz. the time was 8h. 43m. 38$\frac{2}{11}$ sec.

Ex. 2. Let x represent the digit in the place of tens, and y that in the units; then will $10x+y=$ the number itself, and $10y+x$ the number formed by the inverted digits. Hence, by the question,

$$\left.\begin{array}{r} x^2 \backsim y^2=10x+y \\ 10x+y+36=10y+x \end{array}\right\}$$

From the latter, we have $9x-9y=-36$, or $x-y=-4$, or $y=x+4$; whence, it appears, that y is greater than x, and our first equation becomes

$$y^2-x^2=10x+y;$$

Or, by substituting the above value of y, we have

$$(x+4)^2-x^2=10x+x+4$$

That is, $8x+16=10x+x+4$,

or $3x=12$, or $x=4$, and $y=x+4=8$;

Consequently, $10x+y=10\times 4+8=48$, the number sought.

Ex. 3. Let x and y represent the two numbers; then, by the question,

$$\left.\begin{array}{r} x-y : x+y :: 2 : 3 \\ x+y : \ xy \ :: 3 : 5 \end{array}\right\}, \text{ or}$$

$$\left.\begin{array}{l} 3x-3y=2x+2y \\ 5x+5y=3xy \end{array}\right\}$$

From the former of these, $x=5y$, which, substituted for x in the latter, gives

$$25y+5y=15y^2, \text{ or } 30y=15y^2$$

Whence, dividing by $15y$, we have $y=2$, and, consequently, $x=5y=10$.

Ex. 4. Let x be the number of games won, and y the number lost; then, by the question,

$$\left.\begin{array}{r} x+\ y=20 \\ 2x-3y=\ 5 \end{array}\right\}$$

Multiply the first by 3, and we have

$$3x+3y=60$$

Add the latter, $2x-3y=\ 5$

and we have $5x \quad =65$, or $x=13$ games won,

And, consequently, $20-x=20-13=7$ games lost.

Ex. 5. Let x be the number of yards in the four sides; then $3x$ is the number of feet; and, therefore, by the question, $3x-150$ is the number of pallisades.

As is also $x+70$; and, consequently,

$$3x-150=x+70, \text{ or } 2x=220, \text{ or } x=110;$$

Therefore, $3x-150=x+70=180$, the number sought.

Ex. 6. Let x be the number of hours in which B will fill it; then, $\frac{1}{x}$ will be the quantity thrown in by B in an hour, and $\frac{1}{20}$ is the quantity A throws in.

Also, since the two together will fill it in 12 hours, $\frac{1}{12}$ will be the quantity the two throw in in an hour,

$$\text{Whence, } \frac{1}{x}+\frac{1}{20}=\frac{1}{12}, \text{ or}$$

$$\frac{60}{60x}+\frac{3x}{60x}=\frac{5x}{60x}, \text{ or}$$

$$5x-3x=60; \text{ therefore, } 2x=60, \text{ or } x=30. \text{ Ans.}$$

Ex. 7. This is not properly an algebraical question; but the best method of solution is as follows:

Dividing each effect by the time it is produced in, we have $\frac{a}{e}$, $\frac{b}{f}$, and $\frac{c}{g}$, for the momentary effect of each agent, and therefore $\frac{a}{e}+\frac{b}{f}+\frac{c}{g}$, the momentary effect of the three agents; consequently, $d\div(\frac{a}{e}+\frac{b}{f}+\frac{c}{g})$ will be the time in which all three will produce the effect d.

Ex. 8. Let x be the required number; then, by the question, $(x+3):(x+19)::(x+19):(x+51)$.

Or, since by the nature of geometrical progression, the product of the means and extremes are equal to each other, we have

$$(x+3)(x+5)=(x+19)^2,$$
$$\text{or } x^2+54x+153=x^2+38x+361,$$
$$\text{or } 54x-38x=208,$$
$$\text{or } 16x=208, \text{ or } x=13. \quad \text{Ans.}$$

Ex. 9. Let r be the ratio; then, since 3 is the first term, $3 : 3r :: 3r^2 : 24$, are the first four proportionals, and $3r$ and $3r^2$ the two means sought.

Hence, since the product of the extremes and means are equal, we have

$$9r^3=72, \text{ or } r^3=8, \text{ or } r=2;$$

and, therefore, 6 and 12 are the means required.

In the same way, in the second part, we have

$$3 : 3r :: 3r^2 : 3r^3 :: 3r^4 : 96,$$

Whence also we have $3r \times 3r^4 = 3 \times 96$,

$$\text{or } 9r^5=288, \text{ or } r^5=32, \text{ or } r=2,$$

Therefore, 6, 12, 24, and 48 are the four mean proportionals.

Ex. 10. Let x, rx, r^2x, r^3x, r^4x, r^5x be the six proportionals.

Then, by the question,

$$\left.\begin{aligned} x+rx+r^2x+r^3x+r^4x+r^5x&=315 \\ \text{and } x \qquad\qquad\qquad\qquad +r^5x&=165 \end{aligned}\right\}$$

By subtraction, $rx+r^2x+r^3x+r^4x=150$

Now, by the rules for geometrical progression, our equation may be written

$$\left.\begin{aligned} \frac{r^6-1}{r-1}x&=315 \\ \text{and the latter } \frac{r^4-1}{r-1}rx&=150 \end{aligned}\right\}$$

By dividing the former of these by the latter,

$$\frac{r^6-1}{r^4-1}=\frac{315r}{150}=\frac{21r}{10}$$

Reducing the latter, by dividing both terms by r^2-1, we have

$$\frac{r^4+r^2+1}{r^2+1}=\frac{21r}{10},$$

$$\text{or } r^4+r^2+1=\frac{21r}{10}(r^2+1)$$

And by adding r^2 to both sides

$$r^4+2r^2+1=\frac{21r}{10}(r^2+1)+r^2,$$

$$\text{or } (r^2+1)^2-\frac{21r}{10}(r^2+1)=r^2$$

Hence, by completing the square,

$$(r^2+1)^2-\frac{21r}{10}(r^2+1)+\frac{441r^2}{400}=r^2+\frac{441r^2}{400}$$

$$\text{Or } (r^2+1)^2-\frac{21r}{10}(r^2+1)+\frac{441r^2}{400}=\frac{841r^2}{400}$$

And, by extracting the root, $(r^2+1)-\frac{21}{20}r=\frac{29r}{20}$,

$$\text{or } r^2-\frac{25}{10}r=-1$$

$$\text{Or } r=\frac{25}{20}\pm\sqrt{(\frac{625}{400}-1)}=\frac{25}{20}\pm\frac{15}{20}=2\,;$$

But from our first reduced equation we have

$$x=\frac{315(r-1)}{r^6-1}=\frac{315}{63}=5\,;$$

Therefore, 5, 10, 20, 40, 80, and 160, are the proportionals sought.

Ex. 11. Let x and y be the two numbers; then, by the question,

$$\left.\begin{array}{l} x+y=a \\ \frac{1}{x}+\frac{1}{y}=b \end{array}\right\}$$

From the 2d. $x+y=bxy=a$, or $xy=\frac{a}{b}$,

Squaring the 1st, $x^2+2xy+y^2=a^2$

Subtract $\quad 4xy \quad =\frac{4a}{b}$

We have $\quad x^2-2xy+y^2=a^2-\frac{4a}{b}$

Or $\quad x-y=\sqrt{(\frac{a^2b-4a}{b})}$

Repeating again $x+y=a$, the first question,

We have, by addition, $2x=a+\sqrt{(\frac{a^2b-4a}{2})}$,

$$\text{or } x=\frac{1}{2}a+\frac{1}{2}\sqrt{\frac{a}{b}(ab-4)}$$

And by subtraction, $2y=a-\sqrt{(\frac{a^2b-4a}{b})}$,

$$\text{or } y=\frac{1}{2}a-\frac{1}{2}\sqrt{\frac{a}{b}(ab-4)}.$$

Ex. 12. Let x be the number of men employed at first; then, $x+16$ will be the number in the second instance; and since the time in performing the work is reciprocally as the number of men employed, we have

$$x : 16 :: x+16 : 24$$

Whence, $24x=16x+256$, or $x=\frac{256}{8}=32$, the number of men at first, and $32+16=48$, the number during the second part of the time.

$$\text{Hence, } \left.\begin{array}{r}32\times 24\times 1\frac{1}{2}=1152s.\\ 48\times 16\times 1\frac{1}{2}=1152s.\end{array}\right\}=£115\ 4s.\quad \text{Ans.}$$

Ex. 13. Let x and y be the two numbers; then, by the question, we have

$$\left.\begin{array}{r}x^2+y=\ 62\\ y^2+x=176\end{array}\right\}$$

From the first, $y=62-x^2$; and, consequently,

$$y^2=62^2-124x^2+x^4$$

This substituted in the second, gives

$$62^2-124x^2+x^4+x=176$$
$$\text{or } x^4-124x^2=176-3844-x$$
$$\text{or } x^4-124x^2+x=-3668.$$

Whence, x will be found $=7$, and, consequently, $y=62-x^2=13$; but the question cannot, I believe, be in any man-

ner reduced to a quadratic form, at least while it is considered under the general form

$$x^2+y=a, \text{ and } y^2+x=b.$$

Ex. 14. Let x denote the number of feet in the circumference of the less wheel, and y the circumference of the greater wheel.

Then will $\frac{360}{x}$ be the number of the revolutions of the less wheel, and $\frac{360}{y}$ the number of revolutions of the greater.

Whence by the question $\begin{cases} \frac{360}{x}-\frac{360}{y}=6 \\ \frac{360}{x+3}-\frac{360}{y+3}=4. \end{cases}$

From the first, $360(y-x)=6xy$

From the second, $360(y-x)=4(x+3)(y+3)$.

The latter of which, by multiplication, gives

$$360y-360x=4xy+12x+12y+36$$

or $348y-372x=4xy+36$

And the former, by division,

$$60xy-60x=xy$$

Mult. this by 4, $240y-240x=4xy$

Subtract $348y-372x=4xy+36$

And we have $-108y+132x=-36$

Whence $x=\frac{108y-36}{132}=\frac{9y-3}{11}$.

Substitute this value of x in the equation

$$60y-60x=xy,$$

And we have $60y-\frac{540y-180}{11}=\frac{9y^2-3y}{11}$, or

$$660y-540y+180=9y^2-3y, \text{ or}$$

$$9y^2-123y=180, \text{ or}$$

$$y^2-\frac{41}{3}y=20\text{; whence,}$$

$$y=\frac{41}{6}\pm\sqrt{\left(\frac{1681}{36}+20\right)}=\frac{41}{6}\pm\frac{49}{6}=\frac{90}{6}=15$$

and, consequently, $\frac{9y-3}{11}=\frac{135-3}{11}=12$; that is, the circumference of the greater wheel $=15$, and the circumference of the less $=12$.

Ex. 15. Here, by the question,

$$\left.\begin{array}{r} x+y=a \\ mx+ny=b \end{array}\right\}$$

Multiply the first by m, and we have

$$\begin{array}{rl} & mx+my=ma \\ \text{Subtract} & mx+ny=b \\ \hline \end{array}$$

And we have $(m-n)y=ma-b$, or $y=\frac{ma-b}{m-n}$

Again, multiply the first by n, and we obtain

$$\begin{array}{rl} & nx+ny=na \\ \text{Subtract} & mx+ny=b \\ \hline \end{array}$$

And we have $x(n-m)=na-b$; or $x(m-n)=b-na$.

Whence, $x=\frac{b-na}{m-n}$.

Ex. 16. Here, the several portions of wine drawn off were 10 gals. remains 74

$\frac{10.74}{84}$ remains $74-\frac{10.74}{84}=\frac{74^2}{84}$

$\frac{10.74^2}{84^2}$ remains $\frac{74^2}{84}-\frac{10.74^2}{84^2}=\frac{74^3}{84^2}$

$\frac{10.74^3}{84^3}$ remains $\frac{74^3}{84^2}-\frac{10.74^3}{84^3}=\frac{74^4}{84^3}$

Therefore, $\frac{74^4}{84^3}=50.569$ gallons remaining.

Ex. 17. Let x represent the number of persons, and y the number of pounds each received, then xy is the whole sum divided.

Now by the question

$$\left.\begin{array}{r}(x-3)\times(y+150)=xy\\(x+6)\times(y-120)=xy\end{array}\right\}, \text{ or}$$

$$\left.\begin{array}{r}xy-3y+150x-450=xy\\xy+6y-120x-720=xy\end{array}\right\}, \text{ or}$$

$$\left.\begin{array}{r}150x-3y=450\\-120x+6y=720\end{array}\right\}$$

Multiply the first equation by 2, and we have

$$300x-6y=900$$

Add the 2d, $-120x+6y=720$

Whence $180x=1620$, or $x=9$, the number of persons;

And, consequently $y=\dfrac{300x-900}{6}=300$, the sum each received; and $9\times300=2700l.$ the whole sum divided.

Ex. 18. Since the reduced value of the 16 pieces is $8l.$ $8s.$ and the part taken from them is $16\times2s.\ 6d.=2l.$ the original value was $10l.\ 8s.$ or $208s$; consequently $\dfrac{208}{16}=13s.$ the original value of each.

Ex. 19. Let x be the number of ounces of tin, and y the number of ounces of copper.

Then by the question $x+y=505$, and $\dfrac{x}{4\frac{1}{4}}+\dfrac{y}{5\frac{1}{4}}=100$.

The second equation cleared of fractions, is $21x+17y=525\times17$, and the first multiplied by 17,

We have $17x+17y=505\times17$;

Subtracting, gives $4x=20\times17$;

Therefore $x=5\times17=85$, the ounces of tin; and consequently $505-85=420$ ounces of copper.

Ex. 20. Since the privateer gains 2 miles per hour on her prize, and the latter is 18 miles a-head, $\frac{18}{2}=9$ hours the time before she is overtaken; and consequently $9\times8=72$, the distance run.

Ex. 21. Let x represent the number of 7*s*. pieces, and y the number of dollars.

Then by the question

$$7x+4\tfrac{1}{2}y=2000 \text{ shillings, or}$$
$$14x+9\ y=4000;$$

Consequently $y=\frac{4000-14x}{9}=$ whole number,

$$\text{Or } y=444-x+\frac{4-5x}{9}=wh.$$

$$\text{Let now } \frac{4-5x}{9}=p; \text{ then } 9p=4-5x, \text{ or}$$

$$x=\frac{4-9p}{5}=1-2p-\frac{1-p}{5};$$

Assume $\frac{1-p}{5}=q$, and we have $p=-5q+1$;

Consequently $x=\frac{45q-5}{5}=9q-1$, where q may be taken at pleasure.

If we take $q=1,\ 2,\ 3,\ 4,\ 5,$ &c.
$x=8,\ 17,\ 26,\ 35,\ 44,$ &c.

But the greatest value of x cannot exceed $\frac{4000-9}{14}=285$; therefore $\frac{285}{9}=31$, the number of different ways.

Otherwise. By the rule before given, we find from the equation $14p-9q=1$, $p=2$, and $q=3$;

Whence $\frac{2\times4000}{9}-\frac{3\times4000}{14}=31$, the number of different ways; the same as before.

Ex. 22. Let x and y represent the two numbers; then by the question,

$$\left.\begin{array}{l} x+y=2=a \\ x^9+y^9=32=b \end{array}\right\}$$

Assume $xy=p$; then

$$x^3+y^3=a^3-3ap$$
$$x^9+y^9=(a^3-3ap)^3-3p^3(a^3-3ap),$$

Or $a^9-9a^7p+27a^5p^2-30a^3p^3+9ap^4=b$;

Or in numbers, by transposing, &c.

$$p^4-\frac{40}{3}p^3+48p^2-64p+\frac{80}{3}=0,$$

Which equation is resolvable into the factors

$$(p^2-4p+4)(p^2-\frac{28}{3}p+\frac{20}{3})=0$$

Whence by the solution of these two quadratics we have

$$p=2,\ p=-2,\ p=\frac{14}{3}+\frac{\sqrt{136}}{3},\text{ and } p=\frac{14}{3}-\frac{\sqrt{136}}{3};$$

But $x+y=2$, and $xy=p$;
Whence $x-y=\pm\sqrt{(4-4p)}=2\sqrt{(1-p)}$:

And substituting here the above values of p, we have the following solutions, viz.

$$x=1+\sqrt{-1},\text{ and } y=1-\sqrt{-1}$$

$$x=1+\sqrt{\left\{\frac{-11}{3}-\frac{\sqrt{136}}{3}\right\}},\text{ and}$$

$$y=1-\sqrt{\left\{\frac{-11}{3}-\frac{\sqrt{136}}{3}\right\}}$$

$$x=1+\sqrt{\left\{\frac{-11}{3}+\frac{\sqrt{136}}{3}\right\}}=1+\tfrac{1}{3}\sqrt{(6\sqrt{34}-33)}$$

$$\text{and } y=1-\sqrt{\left\{\frac{-11}{3}+\frac{\sqrt{136}}{3}\right\}}=1-\tfrac{1}{3}\sqrt{(6\sqrt{34}-33)},$$

The latter two of which are the only real answers; the others being imaginary.

Ex. 23. Put $a=666$, and $b=406$; then, from the second equation,

$$y=\frac{b-x^3}{x};$$

and this value of y being substituted in the first equation, gives

$$\frac{b^3-3b^2x^3+3bx^6-x^9}{x^3}-b+x^3=a;$$

and multiplying by x^3, and transposing, we have

$$x^9-(3b+1)x^6+(3b^2+b+a)x^3=b^3;$$

or, in numbers, $x^9-1219y^5+937470x^3=406^3$;
from whence, by the resolution of a cubic equation, we have $x^3=343$; therefore

$$x=7\text{, and } y=\frac{b-x^3}{x}=\frac{406-343}{7}=\frac{63}{7}=9\text{, the answer.}$$

Ex. 24. Let x and y be the two numbers; then

The geometrical mean $=\sqrt{xy}$

The arithmetical mean $=\frac{1}{2}x+\frac{1}{2}y$

The harmonical mean $=\dfrac{2xy}{x+y}$

Therefore by the question

$$\tfrac{1}{2}x+\tfrac{1}{2}y-\sqrt{xy}=13$$

$$\sqrt{xy}-\frac{2xy}{x+y}=12;$$

From the first of which equations we have

$$x+y=26+2\sqrt{xy},$$

And, from the 2nd, $x+y=\dfrac{2xy}{\sqrt{xy}-12}$;

Consequently $26+2\sqrt{xy}=\dfrac{2xy}{\sqrt{xy}-12}$;

Whence $26\sqrt{xy}+2xy-312-24\sqrt{xy}=2xy$;

Or $2\sqrt{xy}=312$, or $\sqrt{xy}=156$, and $xy=24336$:

Substituting the value of $\sqrt{xy}$ in the first equation, and repeating our last, we have

$$x+y=338$$
$$xy=24336$$

And by squaring the first of these,

$$x^2+2xy+y^2=114244$$
$$4xy\quad=97344$$

By subtraction, $x^2-2xy+y^2=16900$
By extraction $x-y=130$
Also $x+y=338$;
Whence by addition $2x=468$, or $x=234$
And by subtraction $2y=208$, or $y=104$.

Ex. 25. Here $\left.\begin{array}{l}x^3y+y^3x=3\\x^6y^2+y^6x^2=7\end{array}\right\}$

By squaring the first equation, we have

$$x^6y^2+2x^4y^4+y^6x^2=9$$

Subt. the 2d, $x^6y^2 \qquad +y^6x^2=7$

$$2x^4y^4=2, \text{ or } xy=1.$$

Hence, dividing the first by xy, and the second by x^2y^2, we have $x^2+y^2=3$
$x^4+y^4=7$

From double the last $2x^4+2y^4=14$
Subt. the square of the first $x^4+2x^2y^2+y^4=9$
And we shall have $x^4-2x^2y^2+y^4=5$;
Or by extracting $x^2-y^2=\sqrt{5}$
And by repeating $x^2+y^2=3$
Therefore by addition and subtraction,

$$x^2=\frac{3}{2}+\frac{1}{2}\sqrt{5}, \quad y^2=\frac{3}{2}-\frac{1}{2}\sqrt{5},$$

Whence, extracting these by the rule for binomial surds, we have $x=\frac{1}{2}(5+\sqrt{1})$, and $y=\frac{1}{2}(5-\sqrt{1})$.

Ex. 26. Here the equations are

$$\left.\begin{array}{r}x+z+z=23\\xy+xz+yz=167\\xyz=385\end{array}\right\}\text{ to find } x,\ y, \text{ and } z.$$

From what has been said in the Introduction, relative to the doctrine of equations, it is obvious that these numbers are the co-efficients of a cubic equation which has its three roots equal to the several values of x, y, and z; whence we have at once

$$x^3-23x^2+167x-385=0.$$

The three roots of which equation, by the rules for cubics, are found to be 5, 7, and 11, the numbers sought.

Ex. 27. Here the given equations are

$$xyz=\ \ 231=a$$
$$xyw=\ \ 420=b$$
$$xzw=\ \ 660=d$$
$$yzw=1540=c$$

Multiplying these into each other, we have

$$x^3y^3w^3z^3=\ \ abcd$$

or $x\ y\ w\ z\ =\sqrt[3]{abcd}$

Whence, dividing this last equation by each of the given equations, we have

$$w=\frac{\sqrt[3]{abcd}}{a}=20$$

$$z=\frac{\sqrt[3]{abcd}}{b}=11$$

$$y=\frac{\sqrt[3]{abcd}}{d}=\ 7$$

$$x=\frac{\sqrt[3]{abcd}}{c}=\ 3$$

Ex. 28. Here the equations are

$$x+yz=384=a$$
$$y+xz=237=b$$
$$z+xy=192=c$$

From the first $x=a-yz$; which substituted in the second and third, gives

$$y+az-y\ z^2=b$$
$$z+ay-y^2z\ =c$$

Also from the first of these two equations

$$y=\frac{b-az}{1-z^2}=\frac{az-b}{z^2-1}$$

Whence, substituting this value of y in the latter, we have $$z+\frac{a^2z-ab}{z^2-1}-\frac{z(az-b)^2}{(z^2-1)^2}=c,$$
or $z(z^2-1)^2+a(az-b)(z^2-1)-z(az-b)^2=c(z^2-1)^2$

Which by multiplying and involving the several factors, becomes

$$z^5-cz^4-2z^3+(2c+ab)z^2-(b^2+a^2-1)z=c-ab$$

or, in numbers,

$$z^5-192z^4-2z^3+91245z^2-203624z=-90669.$$

An equation of the 5th degree, the integral root of which is $z=22$; whence $y=\frac{az-b}{z^2-1}=17$, and $x=a-yz=10$.

Ex. 29. Here we have equations

$$x^2+xy=108=a$$
$$y^2+yz=\ \ 69=b$$
$$z^2+zx=580=c$$

Assume $x=my$, and $z=ny$, and these will become

$$y^2(m^2+\ m)=a$$
$$y^2(1\ +\ n)=b$$
$$y^2(n^2\ +nm)=c$$

$$\text{Whence } y^2=\frac{a}{m^2+m}=\frac{b}{1+n}=\frac{c}{n^2+mn}, \text{ or}$$

$$a(1+n)\ \ =b(m^2+m)$$
$$a(n^2+mn)=c(m^2+m)$$

From the first of these we shall have

$$n=\frac{b(m^2+m)-a}{a}.$$

Which value of n, substituted in the second equation, gives $$\frac{\{b(m^2+m)-a\}\times\{bm^2+(b+a)m-a\}}{a}=c(m^2+m),$$

And by reduction,

$$b^2m^4+(ab+2b^2)m^3+(b^2-ab-ac)m^2-a(a+2b+c)m+a^2=0:$$

Or in numbers,

$$m^4+\frac{82}{83}m^3-\frac{65331}{4761}m^2-\frac{89208}{4761}m=\frac{11664}{4761}$$

From which is obtained $m=3$;

Consequently $n=\frac{b(m^2+m)-a}{a}=\frac{20}{3}$, $y=\sqrt{(\frac{a}{m^2+m})}=3$, $x=my=9$, and $z=ny=20$, as required.

Ex. 30. Given $\left.\begin{array}{l} x^2+xy+y^2=5 \\ x^4+x^2y^2+y^4=11 \end{array}\right\}$

Here, dividing the latter by the former, we have

$$x^2-xy+y^2=\frac{11}{5}$$

$$x^2+xy+y^2=5$$

By addition $\quad x^2+y^2=\frac{36}{10}$

By subtraction $\quad xy=\frac{14}{10}$

Also, by adding and subtracting double the latter from the former, we have

$$x^2+2xy+y^2=\frac{36}{10}+\frac{28}{10}=\frac{64}{10}$$

$$x^2-2xy+y^2=\frac{36}{10}-\frac{28}{10}=\frac{8}{10}$$

Whence $x+y=\sqrt{\frac{64}{10}}$

And $\quad x-y=\sqrt{\frac{8}{10}}$

Consequently $x=\frac{1}{2}\sqrt{\frac{64}{10}}+\frac{1}{2}\sqrt{\frac{8}{10}}$,

And $y=\frac{1}{2}\sqrt{\frac{64}{10}}-\frac{1}{2}\sqrt{\frac{8}{10}}$,

Or by reduction, $x=\frac{2}{5}\sqrt{10}+\frac{1}{5}\sqrt{5}$

$$y=\frac{2}{5}\sqrt{10}-\frac{1}{5}\sqrt{5},$$

Which are the values of x and y, as required.

Ex. 31. Here the given equation $x^4-6x^3+13x^2-12x=5$, may be put under the form $(x^2-3x)^2+4(x^2-3x)=5$, (by Note, page 134, Book,) which being now a quadratic, we find $x^2-3x=-2\pm\sqrt{9}=1$, and this is again a quadratic, from which we derive $x=\frac{3}{2}\pm\frac{1}{2}\sqrt{13}$, the answer.

Ex. 32. Let a represent the number of people, and $\frac{a}{x}$ the first year's increase; then at the end of each following year the numbers will be

1st year, $a+\frac{a}{x}$

2d $\quad a+2\frac{a}{x}+\frac{a}{x^2}$

3d $\quad a+3\frac{a}{x}+3\frac{a}{x^2}+\frac{a}{x^3}$

4th $\quad a+4\frac{a}{x}+6\frac{a}{x^2}+4\frac{a}{x^3}+\frac{a}{x^4}$

5th $\quad a+5\frac{a}{x}+10\frac{a}{x^2}+10\frac{a}{x^3}+5\frac{a}{x^4}+\frac{a}{x^5}$

100th $\quad a+m\frac{a}{x}+n\frac{a}{x^2}+p\frac{a}{x^3}+$ &c. $n\frac{a}{x^{99}}+\frac{a}{x^{100}}$

Where m, n, p, &c. are the co-efficients of the binomial $(1+1)^{100}$.

Which expression may therefore be written

$$a(1+\frac{1}{x})^{100},$$

which by the question is to be equal to $2a$; whence

$$(1+\frac{1}{x})^{100}=2, \text{ or } \frac{1}{x}=\sqrt[100]{2}-1;$$

That is, $\frac{1}{x}$=.00695, or $x=\frac{1}{.00695}=144$ nearly.

The annual increase must therefore be $\frac{1}{144}$th part of the population.

Ex. 33. It is obvious that the least number of weights that can be used to weigh 3*lb.* is two, viz. 1*lb.* and 3*lb.* and if to these we add a 9*lb.* we shall be able to weigh all the weights, 9 ± 1, 9 ± 2, 9 ± 3, 9 ± 4, as far as 13*lb.*

Increasing again our weights by 3×9*lb.*=27*lb.* we shall be able to weigh 27 ± 1, 27 ± 2, 27 ± 3, &c. 27 ± 13, that is, to 40*lb.*; and in the same manner, by the addition of three times the last weight, viz. 81, we can weigh 81 ± 1, 81 ± 2, 81 ± 3, 81 ± 4, &c. 81 ± 40.

Therefore 1, 3, 9, 27, 81, &c. are the weights required.*

Ex. 34. Let a, b, x, y, be four numbers, such that $a^2+b^2+x^2=y^2$. Assume $y=x+1$; then $a^2+b^2+x^2=x^2+2x+1$, or $a^2+b^2=2x+1$; hence $x=\frac{1}{2}(a^2+b^2-1)$, where a and b may be taken at pleasure, provided the one be an even, and the other an odd number.

If $a=2$, and $b=3$, then $x=6$, and $y=7$; hence 2, 3, 6, 7, are the least whole numbers that will satisfy the conditions of the question.

If $a=3$, and $b=4$, then $x=12$, and $y=13$, the numbers in the book.

Ex. 35. Let x be the number sought; then by the question $\frac{x-5}{6}$, $\frac{x-4}{5}$, $\frac{x-3}{4}$, $\frac{x-2}{3}$, and $\frac{x-1}{2}$, are to be all whole numbers.

Make $\frac{x-5}{6}=p$; then $x=6p+5$; substitute this in the second, and we have

* The most general method of solving questions of this kind is by means of the ternary scale of notation.—See Barlow's Theory of Numbers, chap. 10.

$\frac{6p+1}{5}=p+\frac{p+1}{5}=wh.$ or $\frac{p+1}{1}=q$, or $p=5q-1$; and consequently $x=30q-1$.

Substitute this in the 3d, and we have

$$\frac{30q-4}{4}=wh. \text{ or } 7q-1+\frac{q}{2}=wh.$$

Whence $q=2r$; consequently $x=60r-1$.

This, substituted in the 4th and 5th equations, gives whole numbers; therefore the general value of $x=60r-1$, and if $r=1$; whence $x=59$, the least value sought.

Ex. 36. Let x be the year required; then if $x+9$, $x+1$, and $x+3$, be divided respectively by 28, 19, and 15, the remainders will be the cycles of the sun, the golden number, and the Roman indiction. Hence

$$\frac{x+9-18}{28}, \frac{x+1-8}{19} \text{ and } \frac{x+3-10}{15}:$$

$$\text{Or, } \frac{x-9}{28}, \frac{x-7}{19} \text{ and } \frac{x-7}{15},$$

must all be whole numbers.

$$\text{Let } \frac{x-9}{28}=p, \text{ then } x=28p+9;$$

and substituting this value in the second equation, it becomes $\frac{28p+2}{19}=wh.$; whence $\frac{28p+2}{19}=p+\frac{9p+2}{19}=wh.$

$$\text{Let } \frac{9p+2}{19}=q; \text{ then } p=\frac{19q-2}{9}=2q+\frac{q-2}{9};$$

$$\text{Consequently } \frac{q-2}{9}=r, \text{ or } q=9r+2;$$

$$\text{Whence } p=\frac{19q-2}{9}=19r+4, \text{ and } x=532r+121.$$

Again, by substitution

$$\frac{x-7}{15}=\frac{532r+114}{15}=35r+7+\frac{2r+9}{15}=wh.$$

$$\text{Or } \frac{2r+9}{15}=s;\ \text{whence } r=\frac{15s-9}{2}=7s-4+\frac{s-1}{2}:$$

$$\text{Assume } \frac{s-1}{2}=t;\ \text{then } s=2t+1.$$

where t might be taken at pleasure; but as the least value is required, let $t=0$, then $s=1$, and $r=3$; and consequently $x=532r+121=1596+121=1717$, the year required.

Ex. 37. Given the equation $256x-87y=1$, to find the least values of x and y, in whole numbers.

By transposition and division.

$$y=\frac{256x-1}{87}=3x-\frac{5x+1}{87}=wh.$$

$$\text{Let } \frac{5x+1}{87}=p;\ \text{then } x=\frac{87p-1}{5}=17p+\frac{2p-1}{5}=wh.$$

$$\text{Assume } \frac{2p-1}{5}=q;\ \text{then } p=\frac{5q+1}{2}=wh.$$

$$\text{Or } \frac{q+1}{2}=wh.=r;\ \text{whence } q=2r-1,$$

where r may be taken at pleasure; if $r=1$, then $q=1$, and $p=3$; whence $x=\frac{87p-1}{5}=52$, and $y=153$, the least numbers that answer the conditions of the equation.

Ex. 38. Let x be one of the equal sides, y the base, and z the perpendicular of the first triangle; and x', y', and z', the corresponding lines in the second triangle.

Then $2x+y$ is the perimeter of the first; and
$2x'+y'$ the perimeter of the second.

Also $\frac{yz}{2}$ the area of the first,

and $\frac{y'z'}{2}$ the area of the second;

Whence, by the question, we must have

$$2x+y=2x'+y'$$

$$\text{and}\quad yz=y'z':$$

Also $x^2=\frac{y^2}{4}+z^2$, and $x'^2=\frac{y'^2}{4}+z'^2$;

Assume $x=r^2+s^2$, and $\frac{y}{2}=r^2-s^2$,

Then $z=(x^2-\frac{y^2}{4})^{\frac{1}{2}}=2rs$.

Again, assuming $x'=r^2+p^2$, and $\frac{y'}{2}=r^2-p^2$, we have $z'=2rp$; and the perimeter of each will be $4r^2$.

Also $\frac{yz}{2}=2rs(r^2-s^2)$, and $\frac{y'z'}{2}=2rp(r^2-p^2)$;

$$\begin{aligned} \text{Whence } 2rs(r^2-s^2)&=2rp(r^2-p^2) \\ \text{or } \quad s(r^2-s^2)&=p(r^2-p^2) \end{aligned}$$

And $r^2=\frac{s^3-p^3}{s-p}=s^2+sp+p^2$

Therefore it remains to find

$$s^2+sp+p^2=\square;$$

Which latter condition has place if we assume

$s=m^2-n^2$, and $p=n^2+2mn$,

In which case $r=m^2+mn+n^2$.

Whence

$$\begin{aligned} x&=r^2+s^2=(m^2+mn+n^2)^2+(m^2-n^2)^2 \\ y&=r^2-s^2=2(m^2+mn+n^2)^2-2(m^2-n^2)^2 \\ x'&=r^2-p^2=(m^2+mn+n^2)^2+(n^2+2mn)^2 \\ y'&=r^2-p^2=2(m^2+mn+n^2)^2-2(n^2+2mn)^2 \end{aligned}$$

Where m and n may be taken at pleasure.

If $m=2$ and $n=1$, then $x=58$, $y=80$, $x'=74$, and $y'=48$, or since all these numbers are divisible by 2, we have

$$\left.\begin{aligned} x&=29,\ y=40 \\ x'&=37,\ y'=24 \end{aligned}\right\}\text{ as required.}$$

Ex. 39. Let x, y, and z represent the base, perpendicular, and hypothenuse of the first triangle.

x', y', and z', those of the second,
and x'', y'', and z'', those of the third;

then, we have to find $\left.\begin{array}{l} x^2+y^2=z^2 \\ x'^2+y'^2=z'^2 \\ x''^2+y''^2=z''^2 \end{array}\right\}$

Also $xy=x'y'=x''y''$.

And, in order to fulfil the three first conditions,

$$\begin{array}{l} \text{Let } x=m^2-n^2, \text{ and } y=2mn \\ x'=m'^2-n'^2, \text{ and } y'=2m'n' \\ x''=m''^2-n''^2, \text{ and } y''=2m''n'' \end{array}$$

Then it remains to find

$$(m^2-n^2)\times 2mn=(m'^2-n'^2)\times 2m'n'$$
$$(m'^2-n'^2)\times 2m'n'=(m''^2-n''^2)\times 2m''n''$$

Which equations may be resolved into the factors

$$(m+n)(m-n)mn=(m'+n')(m'-n')m'n'$$
$$(m''+n'')(m''-n'')m''n''=(m'+n')(m'-n')m'n'$$

Where it is only necessary so to equate the factors of each of these equations, that the reduction of them may give the same ratio to two of the quantities.

For which purpose, let $m+n=2n'$, and $m=m'$

then, $m-n=2(m'-n')$, and $n=2n'-m'$

But, since the product of the preceding factors are equal, we must have

$$2n'-m'=\frac{m'+n'}{4}, \text{ or } 7n'=5m'$$

Again, in the second equation, we may assume

$$m''+n''=3n'$$
$$n''=m'$$
$$\text{and } 2(m''-n'')=m''-n'$$

$$\text{Then, } \tfrac{1}{2}m''=\frac{3n'-m'}{2}$$

Where again, because the product of the factors are equal, we must have

$$\frac{3n'-m'}{2}=\frac{m'+n'}{3}, \text{ or } 5m'=7n',$$

the same ratio as before.

Assuming, therefore, $m'=7$, and $n'=5$, we have

$m=7$, and $n=3$; $m''=8$, and $n''=7$;

Whence, $x=m^2-n^2=40$; $y=2mn=42$

$x'=m'^2-n'^2=24$; $y'=2m'n'=70$

$x''^2=m''^2-n''^2=15$; $y'=2m''n''=112$.

$\therefore z=58$, $z'=74$, $z''=113$.

Ex. 40. Given $x^{\frac{1}{x}}=1.2655$, to find x.

Here, a few trials show that x is between 1.3 and 1.4.

Where, if $x=1.3$, then, $\frac{\log.\ 1.3}{1.3}=.087649$

log. $1.2655=.102262$

Error $-.014613$

And if $x=1.4$, then, $\frac{\log.\ 1.4}{1.4}=.104307$

log. $1.2655=.102262$

Error $+.002115$

Hence, $.01665 : .1 :: .014613 : .0884$

Therefore, $1.3+.0884=1.3884$. Answer.

Ex. 41. This equation is better put under the form $\frac{y^x}{x^y}=.6$,

Or, log. $y\times x$—log. $x\times y$=log. $.6=-1.77815$.

Which, by a few trials, shows us that x is between 4 and 5, and y between 5 and 6; let us, therefore, take one of these quantities, y, as constant, and correct the other, x.

Here, then, assuming $y=5.5$, let us take $x=4.6$ and 4.7; then,

First, if $x=4.6$:

$$\begin{array}{lr} \text{log. } 5.5\times4.6 = & 3.405665 \\ \text{log. } 4.6\times5.5 = & 3.645158 \\ \hline & -1.760507 \\ \text{log. } .6 = & -1.778151 \\ \hline \text{Error} & -\ .017644 \end{array}$$

Secondly, if $x=4.7$, then

$$\begin{array}{lr} \text{log. } 5.5\times4.7 = & 3.479704 \\ \text{log. } 4.7\times5.5 = & 3.696538 \\ \hline & -1.783166 \\ \text{log. } .6 & -1.778151 \\ \hline \text{Error} & +\ .005015 \end{array}$$

Whence .022659 : .1 : : .017644 : .09;

Therefore $x=4.69$ *nearly*.

By a similar process we may find $y=5.51$ nearly; and repeating the operations again on x and y, we have

$x=4.691445$, and $y=5.510132$.

Ex. 42. After a few trials we find $x=4$ nearly, and y a little less than 3.

Let us therefore assume $x=4.01$; then since $4.01^{4.01}=262.18$, we have $y^y=22.82$.

In this exponential equation we may proceed according to the approximating rule given at page 181, from which we determine y to be between 2.8, and 2.9; and by employing these values of y in the second equation, another approximation is found for x, viz. $x=4.0166$.

And employing this again in the same manner, we have $y=2.8257$; with which two values of x and y, repeating the same operations, we find $x=4.016698$, and $y=2.825710$, as required.

Ex. 43. Here, calling $2x$ the number sought, we have to find $2x+1=\square$, and $x+1=\square$.

Assume $x=m^2-2m$; then $x+1=(m-1)^2$ a square, as required; and therefore it remains to find $2m^2-4m+1=\square$ a square.

Assume $1-4m+2m^2=(rm-1)^2$

Then, $2m^2-4m=r^2m^2-2rm$, or $m=\dfrac{2r-4}{r^2-2}$;

Or, since r may be a fraction, let $r=\dfrac{p}{q}$,

Then, $m=\dfrac{2pq-4q^2}{p^2-2q^2}$; where p and q may be taken at pleasure, provided m be integral.

If we take $p=4$, and $q=3$, then $m=6$, and $2x=48$,

$p=7$, and $q=5$, then $m=30$, and $2x=1680$,

the numbers sought; and various others might be found by giving different values to p and q.

Ex. 44. Here x and y being taken to denote the numbers sought, we have to find

$$x^2+y^2=\square$$
$$x^3+y^3=\square$$

Assume $x^2+y^2=(ry-x)^2$; then we have

$$x^2+y^2=r^2y^2-2ryx+x^2, \text{ or } y=r^2y-2rx,$$
$$\text{or } x=\frac{(r^2-1)y}{2r}$$

$$\text{Whence, } x^3+y^3=\frac{(r^2-1)^3y^3}{8r^3}+y^3$$

which is to be a square, or

$$\frac{(r^2-1)^3+8r^3}{2r}y=\square=s^2.$$

$$\text{And, consequently, } y=\frac{2rs^2}{(r^2-1)^3+8r^3},$$

where r and s may be taken at pleasure.

If $r=2$, then $y=\dfrac{4s^2}{91}$, and $x=\dfrac{3s^2}{91}$; so that, taking $s=91$, we have $y=364$, and $x=273$, the numbers required.

Ex. 45. Let x^2 and y^2 be the numbers sought; then, the latter two conditions will be fulfilled, and it will only remain to find

$$x^2+y^2= \text{a cube.}$$

For which purpose, let $x=rz$, let $y=sz$; then,

$$r^2z^2+s^2z^2= \text{a cube.}$$

$$\text{Assume } r^2z^2+s^2z^2=\frac{z^3}{v^3};$$

$$\text{Then, } z= v^3(r^2+s^2)$$
$$x=rv^3(r^2+s^2)$$
$$y=sv^3(r^2+s^2)$$

Where r, s, and v may be assumed at pleasure. If $r=2$, $s=1$, and $v=1$, then $x=10$, and $y=5$; consequently, $x^2=100$, and $y^2=25$, are the numbers sought.

And, by giving different values to r, s, and v, an indefinite number of other answers may be found.

Ex. 46. This is the same question, except a little variation in the enunciation, as Ex. 26 of the Diophantine Problems, where we found generally

$$x=(p^2-q^2),\ y=2pq,\ z=2pq\frac{(p^2-q^2)}{(p^2+q^2)}.$$

Hence, if each of these be multipled by (p^2+q^2), p^4-q^4, $2pq(p^2+q^2)$, and $2pq(p^2-q^2)$ will be the integral roots in the present question; p and q being taken = any unequal numbers whatever.

If $r=2$, and $s=1$; then, 15^4; 12^4, and 20^4 are the biquadrates required.

Ex. 47. Let ax^2, ay^2, and $\frac{ay^4}{x^2}$ represent the three numbers in geometrical progression; then, by the question,

$$(y^2-x^2)a=\square$$
$$\left(\frac{y^4-x^4}{x^2}\right)a=\square$$
$$\left(\frac{y^4-x^2y^2}{x^2}\right)a=\square$$

$$\text{Or } y^2-x^2=am^2$$
$$\text{and } y^4-x^4=an^2$$
$$\text{Whence, } y^2+x^2=\frac{n^2}{m^2}=n'^2.$$

Assume, therefore, $y=p^2-q^2$, and $x=2pq$; where p and q may be taken at pleasure. If $p=2$, and $q=1$, then $y=3$, and $x=4$; and the numbers are

$$16a,\ 9a,\ \text{and } \frac{81a}{16},$$

$$\text{or } 256a,\ 144a,\ \text{and } 81a,$$

a being $=\frac{y^2-x^2}{m^2}$; where m^2 may be any square factor whatever of y^2-x^2.

In the present instance, let $m=1$, then $a=7$, and the required numbers are 1792, 1008, and 567.

Ex. 48. Let x, y, and z denote the three numbers, and assume $x+y=a^2$, $x+z=b^2$, and $y+z=c^2$; then, by subtraction,

$$\left.\begin{array}{l} x-z=a^2-c^2 \\ x-y=b^2-c^2 \\ y-z=a^2-b^2 \end{array}\right\}\text{all squares.}$$

We have, therefore, only to find such values of a, b, and c, as will satisfy the latter conditions; for in that case the former must have place.

But such values of a, b, and c, have been found in Ex. 27, Diophantine Problems, viz. $697^2=485809$, $185^2=34225$, and $153^2=23409$.

Hence, considering these quantities as known, we have, by the common rules,

$$x=\frac{a^2+b^2-c^2}{2}=248312\tfrac{1}{2}$$

$$y=\frac{a^2+c^2-b^2}{2}=237496\tfrac{1}{2}$$

$$z=\frac{-a^2+b^2+c^2}{2}=214087\tfrac{1}{2}\cdot$$

Or, multiplying each by 4, in order to avoid fractions, we have $x=993250$, $y=949986$, and $z=856350$.

Which numbers answer the conditions of the question; and various others may be had by finding different values for a, b, and c.

Ex. 49. Let x and y be the numbers sought; then, we have to find

$$x+y=a \textit{ square},$$
$$x^2+y^2=a \textit{ biquadrate}.$$

First, in order to make $x^2+y^2=$ a square, assume

$$x=\pm p^2-q^2, \text{ and } y=2pq,$$

Then shall $\quad x^2+y^2=(p^2+q^2)^2$

But when $x^2+y^2=$ a biquadrate, p^2+q^2 must be a square; assume, therefore, again,

$$p=r^2-s^2, \text{ and } q=2rs,$$

Then we shall have

$$x^2+y^2=(p^2+q^2)^2=(r^2+s^2)^4, \textit{ a biquadrate}, \text{ as required.}$$

And it now only remains to find

$$x+y=r^4+4r^3s-6r^2s^2-4rs^3+s^4=\square$$

Hence, in order to reduce this to a more convenient form for solution, substitute $r=\frac{3s}{2}+t$, and the above formula, after multiplying by 16, reduces to

$$s^4+296s^3t+408s^2t^2+160st^3+16t^4=\square$$

Again, assume the formula

$$=(s^2+148st-4t^2)^2=s^4+296s^3t+21896s^2t^2-1184st^3+16t^4,$$

Then, by cancelling the like terms in both, this reduces to

$$21896s-1184t=408s+160t;$$

$$\text{Whence, } \frac{s}{t}=\frac{1344}{21488}=\frac{84}{1343}$$

Assume, therefore, $s=84$, and $t=1343$, and we shall have

$$r=\frac{3s}{2}+t=1469, \text{ and, consequently,}$$

$x=r^4-6r^2s^2+s^4=4565486027761$

$y=4r^3s-4rs^3=1061652293520$, for the numbers sought.

Ex. 50. The solution of this question is intimately connected with that of Ex. 48; for if we here call w, x, y, and z, the four numbers, and at the same time make $w=x+y+z$, we shall have to find

$$w-x=y+z=\square$$
$$w-y=x+z=\square$$
$$w-z=x+y=\square$$

Also $x-y=\square$, $x-z=\square$, and $y-z=\square$.

It is, therefore, only necessary that x, y, and z, may be such numbers that the sum and difference of every two of them may be a square, which are the conditions of Ex. 48, where we found the three numbers to be

$$x=993250$$
$$y=949986$$
$$z=856350$$

And consequently $w=2799586$

Which numbers answer the conditions of the question.

And if, in our 48th Example, we had taken $a=2165$, $b=2067$, and $c=2040$, we should have found

$$x=2399057$$
$$y=2288168$$
$$z=1873432$$
and $w=6560657$

which are the numbers given in the answer in the Introduction.

Ex. 51. Here the proposed series may be put under the form

$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}, \text{ \&c.} = \frac{3}{2}\times\frac{1}{3}$$
$$+\frac{1}{9}+\frac{1}{27}+\frac{1}{81} \text{ \&c.} = \frac{3}{2}\times\frac{1}{9}$$
$$+\frac{1}{27}+\frac{1}{81} \text{ \&c.} = \frac{3}{2}\times\frac{1}{27}$$
$$\text{\&c.} = \frac{3}{2}\times\frac{1}{81}$$

Whence, by the rules and formulæ for geometrical progression, the whole sum is

$$=\frac{3}{2}\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{18}+, \&c.\right)=\frac{3}{2}\times\frac{3}{2}\times\frac{1}{3}=\frac{3}{4}$$

Ex. 52. This may be separated into two series

$$\frac{3}{4}+\frac{27}{64}+\frac{243}{1024}+, \&c. =\frac{16}{7}\times\frac{3}{4}$$

$$-\left(\frac{9}{16}+\frac{81}{256}+\frac{729}{4096}+, \&c.\right)=\frac{16}{7}\times\frac{9}{16}$$

Whence, the whole series $=\frac{12}{7}-\frac{9}{7}=\frac{3}{7}$. Answer.

Ex. 53. Here, by the rules for arithmetical progression, the nth term $=5+(n-1)$.

Therefore, $\{10+(n-1)\}\frac{n}{2}=\frac{n}{2}(n+9)$ is the sum required.

Ex. 54. The 25th term of the progression

1, 2, 4, 8, 16, &c.

$=2^{24}=16777218$; therefore, the 25th term of the proposed series is 216777216; that is, the 16777216th power of 2.

Now, the log. of 2	=0.3010300
Mult. by	16777216
Gives	5050445.3324800

for the logarithm of the 25th term; consequently, the index being 5050445, the number of integers will be 5050446.

Ex. 55. This series is the same as

$$(2^2-2)+(4^2-4)+(6^2-6)+, \&c. \text{ viz.}$$

$$\left\{\begin{matrix} 4(1^2+2^2+3^2+4^2+, \&c.\ 100^2) - \\ 2(1+2+3+4+, \&c.\ 100) \end{matrix}\right\}$$

But the sum of the former

$$=4\times\frac{(n+1)n(2n+1)}{6}=\frac{4\times101\times100\times201}{6}=1353400,$$

And the sum of the latter $=2\times\frac{101\times100}{2}=10100,$

Whence, $1353400-10100=1343300$, the sum required.

Ex. 56. Here, the general form of the series being

$$\frac{(n+1)(n)(2n+1)}{6}$$

and n being $=50$, we shall have

$$\frac{51\times50\times101}{6}=42925, \text{ the sum required.}$$

Ex. 57. By the differential formula, we have

35,	72,	111,	152,	&c.	
	37,	39,	41,	&c.	1st. diff.
		2,	2,	&c.	2d diff.

Whence, $a=35$, $d'=37$, $d''=2$, $n=25$;

And, consequently,

$$na+\frac{n(n-1)}{1.2}d'+\frac{n(n-1)(n-2)}{1.2.3}d''=$$

$$35\times25+12\times25\times37+25\times8\times23=875+11100+4600$$

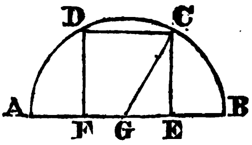
$=16575$, the sum sought.

APPLICATION OF ALGEBRA TO GEOMETRY.

MISCELLANEOUS PROBLEMS.

PROBLEM I.

Let ABCD, be the given semicircle : AB, its diameter ; G, it centre : and CDFE, the required square.



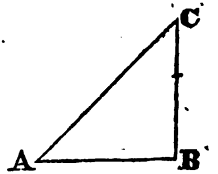
Then, since DF=CE, we have FG=GE.

Let therefore AB$=d$, or CG$=\frac{1}{2}d$; also CE$=x$, and consequently GE$=\frac{1}{2}x$; then, (by Euc. 1. 47,)

$$CG^2=GE^2+CE^2, \text{ or } x^2+\tfrac{1}{4}x^2=\tfrac{1}{4}d^2 ;$$

$$\text{Whence } 5x^2=d^2, \text{ or } x=d\sqrt{\tfrac{1}{5}}=\tfrac{1}{5}d\sqrt{5}.$$

2. Let ABC be the required right angled triangle; in which take AC$=13=h$, AB$=x$, BC$=y$, and the given difference AB$\backsim$BC$=7=d$;



Then, by Euc. 1. 47, we shall have

$$x^2+y^2=h^2$$
$$x-y=d$$

Where squaring the second equation, and subtracting it from double the first, we have

$$x^2+2xy+y^2=2h^2-d^2$$

Consequently, $x+y=\sqrt{(2h^2-d^2)}$

and $x-y=d$

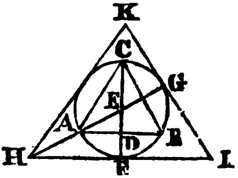
Whence, $x=\frac{1}{2}\sqrt{(2h^2-d^2)}+\frac{1}{2}d$

and $y=\frac{1}{2}\sqrt{(2h^2-d^2)}-\frac{1}{2}d$

Substituting here $h=13$, $d=7$, we have $x=12$, and $y=5$, as required.

Ex. 3. Let AFGC be the given circle, and ABC the required inscribed equilateral triangle.

Join A and the centre E; also join CE, and produce it to D.



Then, by Euc. (IV. 2,) the angle D is a right angle, and the triangles ADE and ADC are similar.

But AD$=\frac{1}{2}$AC, therefore also DE$=\frac{1}{2}$AE.

Let, then, AE=radius$=\frac{1}{2}d$; and, consequently, ED$=\frac{1}{2}$AE $=\frac{1}{4}d$; also put AB$=x$, or AD$=\frac{1}{2}x$.

Then, by Euc. (I. 47,) $\frac{1}{4}x^2=\frac{1}{4}d^2-\frac{1}{16}d^2=\frac{3}{16}d^2$;

Whence, $x=\sqrt{\frac{3}{4}d^2}=\frac{1}{2}d\sqrt{3}$, the side of the inscribed triangle.

Again, produce CD to F, and AE both ways to G and H; and draw HI, IK, perpendicular to EF and EG.

Then, it is obvious, from the propositions above referred to, and Euc. (IV. 3,) that EF$=\frac{1}{2}$HE, and HF$=\frac{1}{2}$HI.

Let, now, HI (the side of the circumscribed triangle) $=y$, and we shall have

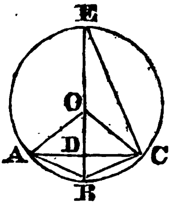
HE$^2=$EF$^2=$HF^2; and, since EF$=\frac{1}{2}d$, HE$=d$,

and the above becomes

$$d^2=\frac{1}{4}d^2+\frac{1}{4}y^2\text{, or } y^2=3d^2\text{, or } y=d\sqrt{3}.$$

the side of the circumscribing triangle.

Ex. 4. It appears, from Euc. IV. 10, that the side of an equilateral and equiangular pentagon inscribed in a circle, is found by dividing the radius of the circle into extreme and mean ratio, the greater part of which is the side of the decagon.



Hence, calling the radius OB$=r$, and the greater part OD $=x$, we must have $r(r-x)=x^2$, or $x^2+rx=r^2$;

Whence $x=-\frac{1}{2}r+\frac{1}{2}\sqrt{5r^2}$, or $x=\frac{1}{2}r(-1+\sqrt{5})$

That is, BC or AB, in the above figure, $=\frac{1}{2}r(-1+\sqrt{5})$

Produce BO to E, and join EC; then, by Euc. I. 47,

$EC^2=EB^2-BC^2=r^2(\frac{5}{2}+\frac{1}{2}\sqrt{5})$, or $EC=r\sqrt{(\frac{5}{2}+\frac{1}{2}\sqrt{5})}$

Again, as $EB : EC :: EC : ED=r(\frac{5}{4}+\frac{1}{4}\sqrt{5})$,

and $ED : CB :: BC : BD=r(\frac{3}{4}-\frac{1}{4}\sqrt{5})$,

Whence, $DC=\sqrt{(ED\times DB)}=\sqrt{\{r^2(\frac{5}{4}+\frac{1}{4}\sqrt{5})(\frac{3}{4}-\frac{1}{4}\sqrt{5})\}}=$ $\frac{1}{4}r\sqrt{(10-2\sqrt{5})}$, or $AC=\frac{1}{4}d\sqrt{(10-2\sqrt{5})}$.

the side of the pentagon required.

Ex. 5. Let $x=$ the length of the rectangle,

and $y=$ the breadth;

then, $2x+2y=$ perimeter, and $xy=$ area.

Now, since the side of the square $=a$; its perimeter $=4a$, and its area $=a^2$, we have

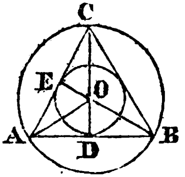
$$\left.\begin{array}{r} xy=\frac{1}{2}a^2 \\ 2x+2y=4a, \end{array}\right\}$$

Whence, $x^2+2xy+y^2=4a^2$
Subtract $\quad 4xy \quad =2a^2$
We have $x^2-2xy+y^2=2a^2$,
or $\quad x-y=a\sqrt{2}$
But $\quad x+y=2a$
Whence, by addition and subtraction,

$$x=a+\tfrac{1}{2}a\sqrt{2}=a(1+\tfrac{1}{2}\sqrt{2})$$
$$y=a-\tfrac{1}{2}a\sqrt{2}=a(1-\tfrac{1}{2}\sqrt{2})$$

the length and breadth as required.

Ex. 6. Let ABC be the given equilateral triangle, and bisect the two sides AB, AC, by the two perpendiculars DO, EO:



Then shall the point O be the centre of the inscribed circle; and OD its radius.

Also, if AO, OB, be joined, they will bisect the angles A and B, and O will therefore be the centre of the circumscribed circle, and OA its radius, Euc. (IV. 4) and (IV. 5.)

Again, if OD be produced, it follows from these propositions, that it will pass through C, and bisect the angle ACB.

Therefore, the triangles ACD and AOD are similar; and since $AD=\frac{1}{2}AC$, therefore $DO=\frac{1}{2}AO$.

Let now $AB=s$, the given side, or $AD=\frac{1}{2}s$; also, $DO=x$, and consequently $AO=2x$; then, $AO^2=AD^2+DO^2$,

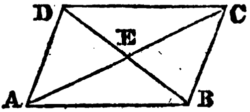
$$\text{or } 4x^2=\tfrac{1}{4}s^2+x^2\,;$$

$$\text{whence, } 3x^2=\tfrac{1}{4}s^2, \text{ or } x=\tfrac{1}{2}s\sqrt{\tfrac{1}{3}}=\tfrac{5}{3}\sqrt{3}=2.8868,$$

$$\text{and } 2x=s\sqrt{\tfrac{1}{3}}=\tfrac{10}{3}\sqrt{3}=5.7736\,:$$

the two radii required.

Ex. 7. Let ABCD be the rhombus, and AC, DB, its two diagonals, intersecting each other in E.



Also, let the perimeter $=4p$, that is, each side of the rhombus $=p$, and the sum of the two diagonals $=s$.

Then, since the diagonals of parallelograms bisect each other, $CE+EB=\frac{1}{2}s$.

And because the three sides of the triangle DEC, and CEB, are respectively equal to each other, the angles at E are each equal to a right angle.

Therefore, calling $AC=x$, and $DB=y$, or $CE=\frac{1}{2}x$, and $EB=\frac{1}{2}y$, we have

$$\left.\begin{array}{r} x+y=s \\ \frac{1}{4}x^2+\frac{1}{4}y^2=p^2 \end{array}\right\}$$

From 8 times the latter,

$$2x^2+2y^2=8p^2$$

Subtract $(x+y)^2=x^2+2xy+y^2=s^2$

And we have $\quad x^2-2xy+y^2=8p^2-s^2$

or $\quad x-y=\sqrt{(8p^2-s^2)}$

But $\quad x+y=s$;

Whence, by addition and subtraction,

$$x=\tfrac{1}{2}s+\tfrac{1}{2}\sqrt{(8p^2-s^2)}$$
$$y=\tfrac{1}{2}s-\tfrac{1}{2}\sqrt{(8p^2-s^2)}$$

Or, since $s=8$, and $p=3$; these become

$$x=4+\tfrac{1}{2}\sqrt{8}=4+\sqrt{2}$$
$$y=4-\tfrac{1}{2}\sqrt{8}=4-\sqrt{2}$$

Which are the two diagonals required.

Ex. 8. Here the three sides of the right angled triangle are x^{3x}, x^{2x}, x^{x}; and, since the square of the longest side is equal to the sum of the squares of the other two, (Euc. I. 47,) we have

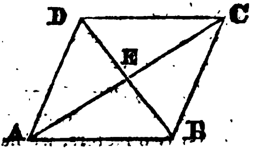
$$x^{6x}=x^{4x}+x^{2x}\text{ ; or }x^{4x}-x^{2x}=1$$

$$\text{Whence, }x^{2x}=\tfrac{1}{2}+\tfrac{1}{2}\sqrt{5}=1.618034$$

$$\text{And }x^{x}=\sqrt{1.618034}=1.272020\text{ ;}$$

$$\text{Therefore, }\frac{x^{2x}\times x^{x}}{2}=1.029085\text{, the area.}$$

Ex. 9. It is a well known geometrical theorem, that the diagonals of a parallelogram bisect each other; and that the sum of their squares is equal to the sum of the squares of the four sides of the parallelogram.



If, therefore, we represent the given parallelogram by the figure ABCD, and make its side BC$=a$, DC$=b$, DB$=d$, and AC$=x$, we have from the above theorem

$$x^2+d^2=2a^2+2b^2,$$

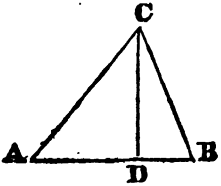
$$\text{or }x^2=2a^2+2b^2-d^2,$$

$$\text{or }x=\sqrt{(2a^2+2b^2-d^2)}$$

Which is the diagonal required.

Ex. 10. Let ABC be the proposed triangle, CD its perpendicular, and AD, DB the segments, of which the difference is given.

Let that difference $=d$, the sum of the sides $=s$, and perpendicular CD$=p$.



Also, put AD$=y+\frac{1}{2}d$, and DB$=y-\frac{1}{2}d$; then, (Euc. I. 7. 4)

$$\sqrt{\left\{(y+\tfrac{1}{2}d)^2+p^2\right\}}=\text{AC},$$
$$\sqrt{\left\{(y-\tfrac{1}{2}d)^2+p^2\right\}}=\text{CB};$$

And, by the question,

$$\sqrt{\left\{(y+\tfrac{1}{2}d)^2+p^2\right\}}+\sqrt{\left\{(y-\tfrac{1}{2}d)^2+p^2\right\}}=s;$$

Squaring both sides, and transposing

$$2\sqrt{\left\{(y+\tfrac{1}{2}d)^2+p^2\right\}}\times\sqrt{\left\{(y-\tfrac{1}{2}d)^2+p^2\right\}}=s^2-2y^2-\tfrac{1}{2}d^2-2p^2;$$

Or, in order to simplify the expression, let

$$s^2-\tfrac{1}{2}d^2-2p^2=2m;\text{ then,}$$
$$\sqrt{\left\{(y+\tfrac{1}{2}d)^2+p^2\right\}}\times\sqrt{\left\{(y-\tfrac{1}{2}d)^2+p^2\right\}}=m-y^2$$

Squaring again both sides, and actually performing the multiplication of the first two factors, we have

$$(y^2-\tfrac{1}{4}d^2)^2+2p^2(y^2+\tfrac{1}{4}d^2)+p^4=m^2-2my^2+y^4;$$

And, by involving and collecting the terms,

$$\tfrac{1}{16}d^4-\tfrac{1}{2}d^2y^2+2p^2y^2+\tfrac{1}{2}p^2d^2+p^4=m^2-2my^2,$$
$$(2m+2p^2-\tfrac{1}{2}d^2)y^2=m^2-\tfrac{1}{2}p^2d^2-\tfrac{1}{16}d^4-p^4,$$
$$\text{Whence } y=\sqrt{\left(\frac{m^2-\frac{1}{2}p^2d^2-\frac{1}{16}d^4-p^4}{2m+2p^2-\frac{1}{2}d^2}\right)}$$

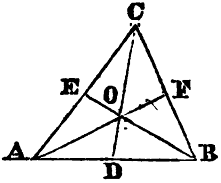
In which formula, substituting the values of p, d, and m, we have $y=472\frac{1}{2}$;

$$\text{Whence, } y+\tfrac{1}{2}d=720=\text{AD},$$
$$y-\tfrac{1}{2}d=225=\text{DB},$$

$$\text{Consequently, }\begin{cases}\text{AC}=\sqrt{(\text{AD}^2+\text{CD}^2)}=780\\ \text{CB}=\sqrt{(\text{DB}^2+\text{CD}^2)}=375\\ \text{AB}=\text{AD}+\text{DB}=945,\end{cases}$$

Which are the three sides required.

Ex. 11. Let ABC be the required triangle, and AF, BE, and CD, the three given lines bisecting the three sides, CB, AC, and AB.



Make $AF=a$, $BE=b$, $CD=c$; also $CB=x$, $AC=y$, and $AB=z$.

Now it is a well known property of triangles, that "double the square of a line drawn from any angle of a triangle to the opposite side, together with double the square of half that side, is equal to the sum of the squares of the other two sides;" that is,

$$2a^2+\tfrac{1}{2}x^2=y^2+z^2$$
$$2b^2+\tfrac{1}{2}y^2=x^2+z^2$$
$$2c^2+\tfrac{1}{2}z^2=x^2+y^2$$

Or,

$$y^2+z^2-\tfrac{1}{2}x^2=2a^2,$$
$$-\tfrac{1}{2}y^2+x^2+z^2=2b^2, \text{ and}$$
$$x^2+y^2-\tfrac{1}{2}z^2=2c^2.$$

From whence, by taking the former of these equations from twice the sum of the two latter, there comes out

$$4x^2+\tfrac{1}{2}x^2=2(2b^2+2c^2-a^2);$$

And consequently

$$x=\tfrac{2}{3}\sqrt{(2b^2+2c^2-a^2)}.$$

In like manner

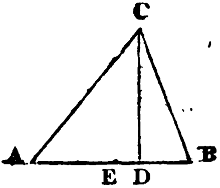
$$y=\tfrac{2}{3}\sqrt{(2a^2+2c^2-b^2)}, \text{ and } z=\tfrac{2}{3}\sqrt{(2a^2+2b^2-c^2)};$$

Where, by substituting the given values of a, b, and c, viz. $a=18$, $b=24$, $c=30$, we have

$$x=34.176,\ y=28.844, \text{ and } z=20,$$

which are the sides required.

Ex. 12. Let ABC be the proposed triangle, of which the base AB is given $=50=2b$.



Then, since the area is also given $=796$, the perpendicular $=\frac{796}{25}=p$, is also known.

Make, now, AE= half base $=a$, and CD$=p$, and ED$=x$; then AD$=a+x$, and BD$=a-x$; AC$=\sqrt{\{p^2+(a+x)^2\}}$, and BC$=\sqrt{\{p^2+(a-x)^2\}}$. Whence, calling the given difference $=10=d$, we have

$$\sqrt{\{p^2+(a+x)^2\}}-d=\sqrt{\{p^2+(a-x)^2\}},$$

Which equation, squared, gives

$$p^2+(a+x)^2-2d\sqrt{\{p^2+(a+x)^2\}}+d^2=p^2+(a-x)^2\ ;$$

And this, by reduction, becomes

$$4ax+d^2=2d\sqrt{\{p^2+(a+x)^2\}}.$$

And this, again squared, produces

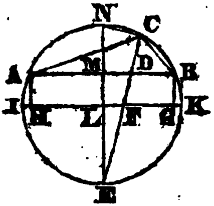
$$16a^2x^2+8ad^2x+d^4=4d^2\times(p^2+a^2+2ax+x^2)\ ;$$

$$\text{or } 16a^2x^2+d^4=4d^2\times(a^2+p^2)+4d^2x^2.$$

$$\text{Whence, } x=\sqrt{\left(\frac{4d^2\times(a^2+p^2)-d^4}{16a^2-4d^2}\right)}.$$

Where, by substituting the numeral values for a, p, and d, the answers for the sides will be found.

Ex. 13. Let ABC be the proposed triangle, AB its base $=194=b$; IK the diameter of the circumscribing circle $=200=d$, drawn parallel to AB; and DC the bisecting line $=66=a$.



Then we shall have $HI=GK=\frac{1}{2}(IK-AB)=3$.

And consequently

$AH=GB=\sqrt{(IG\times GK)}=\sqrt{197\times 3}=\sqrt{591}=c$.

Let now CD be produced to meet the circle in E;

Then, because CD bisects the angle ACB, it will bisect the arc AEB, and therefore the perpendicular ELM will pass through the centre L;

Consequently $EM=100+\sqrt{591}=e$ is also known, as is also $MN=100-\sqrt{591}=f$.

Now let $DE=x$; then, since the two triangles ENC and MDE are similar, we have

$$ME : DE :: CE : NE, \text{ or}$$
$$e : x :: a+x : d;$$

Whence $x^2+ax=de$, or $x=-\frac{a}{2}+\sqrt{(\frac{a^2}{4}+de,)}$ which thus becomes known; and consequently the rectangle $CD\times DE$,

or $a\times\left\{\frac{-a}{a}+\sqrt{(\frac{a^2}{4}+de)}\right\}=r$, it also known.

But this latter rectangle $CD\times DE=AD\times DB$; therefore $AD\times DB$, and $AD+DB$, are both known.

Assume now, $AD=y$, and $BD=z$, then we have

$$\left.\begin{matrix} y+z=b \\ yz=r \end{matrix}\right\}$$

Whence is readily found

$$y=\tfrac{1}{2}b+\tfrac{1}{2}\sqrt{(b^2-4r^2)}=m$$
$$z=\tfrac{1}{2}b-\tfrac{1}{2}\sqrt{(b^2-4r^2)}=n$$

X

Again, calling AC$=v$, and BC$=u$, and we have

$$v : u :: m : n,$$

$$\text{or } v=\frac{um}{n};$$

$$\text{also, } uv=a^2+mn.$$

Substituting $\frac{um}{n}$ for v in the second equation, gives

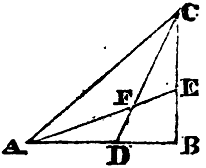
$$\frac{mu^2}{n}=a^2+mn.$$

Whence, $u=\sqrt{(\frac{a^2n+mn^2}{m})}$, and, consequently,

$v=\frac{um}{n}=\sqrt{(\frac{a^2m+m^2}{n})}$, the sides required.

The numerical values of which may be found by substituting those of a, b, and d, in the original equations.

Ex. 14. Let ABC be the proposed triangle, AE$=a$, and DC$=b$, the two given lines.



Also, let x and y represent the sine and cosine of the angle BAC respectively; then, by trigonometry, we have

$\sqrt{(\frac{1+y}{2})}=cos.$ BAE, and $\sqrt{(\frac{1+x}{2})}=cos.$ BCD.

Also, $x : b :: \sqrt{(\frac{1+x}{2})} : \frac{b}{x}\sqrt{(\frac{1+x}{2})}=$AC,

and $y : a :: \sqrt{(\frac{1+y}{2})} : \frac{a}{y}\sqrt{(\frac{1+y}{2})}=$AC;

Whence, $\frac{b}{x}\sqrt{(\frac{1+x}{2})}=\frac{a}{y}\sqrt{(\frac{1+y}{2})}$, or

$$\sqrt{(\frac{1+x}{1+y})}=\frac{ax}{by}.$$

Again, by trigonometry, $sin\ A$*$=\frac{2tan\frac{1}{2}A}{1+tan^2\frac{1}{2}A}$, and

$$cos\ A=\frac{1-tan^2\frac{1}{2}A}{1+tan^2\frac{1}{2}A};$$

Putting, therefore, $tan BAE=t$, and substituting

$$x=\frac{2t}{1+t^2},\text{ and } y=\frac{1-t^2}{1+t^2},$$

We have $\frac{1+t}{\sqrt{2}}=\frac{2at}{b(1-t^2)}$, or

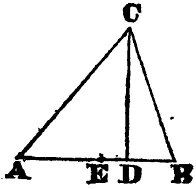
$$b(1+t)(1-t^2)=2at\sqrt{2}, \text{ or}$$

$$t^3+t^2+(\frac{1+2a\sqrt{2}}{b})t=1.$$

Which is a cubic equation, whence the value of t may be determined; viz. the tangent of the angle BAE: and

Hence, also, the angles BDC and BEA become known, and, consequently, the sides AB=35.80737, BC=47.40728, and AC=59.41143, as required.

Ex. 15. Let ABC be the proposed triangle.



* A denotes any angle; but, in this example, it is put for the angle BAC. See Gregory's Trig.

Then the base AB$=8$, or AE$=\frac{1}{2}$ base $=4=b$, CD$=4=p$, and AC$+$BC$=12=s$; and make also ED$=x$.

Then AD$=b+x$, and DB$=b-x$; and consequently

$$\sqrt{\{(b+x)^2+p^2\}}=AC$$

$$\sqrt{\{(b-x)^2+p^2\}}=BC;$$

Whence, by the question

$$\sqrt{\{(b+x)^2+p^2\}}+\sqrt{\{(b-x)^2+p^2\}}=s$$

Squaring both sides, and transposing

$$2\sqrt{\{(b+x)^2+p^2\}}\times\sqrt{\{(b-x)^2+p^2\}}=s^2-2b^2-2x^2-2p^2;$$

Or, in order to simplify, writing $s^2-2b^2-2p^2=2m$

$$\sqrt{\{(b+x)^2+p^2\}}\times\sqrt{\{(b-x)^2+p^2\}}=m-x^2.$$

Squaring again, and collecting the terms

$$(b^2-x^2)^2+2p^2(b^2+x^2)+p^4=m^2-2mx^2+x^4,$$

$$\text{Or } b^4-2b^2x^2+2p^2b^2+2p^2x^2+p^4=m^2-2mx^2;$$

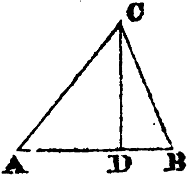
$$\text{Therefore } x^2(2p^2-2b^2+2m)=m^2-2p^2b^2-b^4-p^4$$

$$\text{Or } x=\sqrt{\left(\frac{m^2-2p^2b^2-b^4-p^4}{2p^2-2b^2+2m}\right)}$$

Whence x, and consequently AD and DB become known, and hence also AC$=\sqrt{(AD^2+CD^2)}$ and CB$=\sqrt{(DB^2+CD^2)}$, are determined.

In the present case, $p=4$, $b=4$, $m=40$, whence $x=\frac{6}{5}\sqrt{5}$; therefore AC$=6+\frac{4}{5}\sqrt{5}$, and BC$=6-\frac{4}{5}\sqrt{5}$.

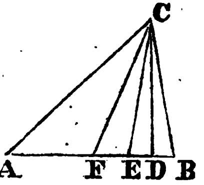
Ex. 16. Let ABC be the proposed triangle.



Let AB$=a$, CD$=\frac{45}{7\frac{1}{2}}=b$, and AD$=x$; therefore BD$=a-x$; also the ratio AC : CB : : 3 : 2, or $m:n$; then, AC$^2=b^2+x^2$, and BC$^2=b^2+(a-x)^2$; consequently $b^2+x^2:b^2$

$+(a-x)^2 :: m^2 : n^2$. Whence, we have $m^2b^2+m^2a^2-2m^2ax+m^2x^2=n^2b^2+n^2x^2$, or $(m^2-n^2)x^2-2m^2ax=(n^2-m^2)b^2-m^2a^2$; therefore, by solving this quadratic, and substituting the values of a, b, m, and n, the numeral value of x may be determined, and hence those of AC and BC.

Ex. 17. Let ABC be the proposed triangle, and make the perpendicular CD$=24=p$, CE the line bisecting the angle ACB$=25=b$, and CF, the line bisecting the base, $=40=c$:



Then, (Euc. I. 47) ED$=\sqrt{(\text{CE}^2-\text{CD}^2)}=7=m$,

Also, FD$=\sqrt{(\text{FC}^2-\text{CD}^2)}=32=n$;

And, in order to simplify, let EF$=q$.

Also, let half the base AF$=$FB$=x$; then,

AE$=x+q$, EB$=x-q$, AD$=x+n$, DB$=x-n$;

$$\text{Hence, AC}=\sqrt{\{(x+n)^2+p^2\}}$$
$$\text{BC}=\sqrt{\{(x-n)^2+p^2\}}$$

And, from (Euc. VI. 3,) we have

AC : BC :: AE : EB,

$$\text{or } \sqrt{\{(x+n)^2+p^2\}} : \sqrt{\{(x-n)^2+p^2\}} :: x+q : x-q;$$

$$\text{Whence } \{(x+n)^2+p^2\}\times(x-q)^2=\{(x-n)^2+p^2\}\times(x+q)^2$$

Which, by multiplying, cancelling, &c. becomes

$$nx(x^2+q^2)=qx(x^2+n^2+p^2)$$

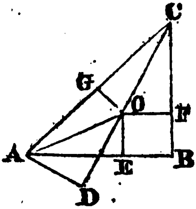
$$\text{Where } x^2=\frac{qn^2+qp^2-nq^2}{n-q}, \text{ or}$$

$$2x=2\sqrt{\left(\frac{qn^2-nq^2+qp^2}{n-q}\right)},$$

the base of the triangle; which, by substituting the proper

numeral values of q, n, and p, gives $\frac{250}{7}\sqrt{14}$; from which and the given lines the other two sides are readily obtained.

Ex. 18. Let ABC be the proposed right angled triangle, and O the centre of its inscribed circle; and let CO—AO $=2=d$, and $AC=10=h$.



Produce CO to D, and let it fall upon the perpendicular AD; which put $=x$.

Then, since CO and AO bisect the two angles C and A, and these two angles together are equal to a right angle, it follows that the two angles OAC and OCA= half a right angle.

But the outward $\angle$ of any triangle, being equal to the two inward opposite $\angle$ s, $\angle$ AOD$=\angle$OAC$+\angle$OCA.

Whence, also AOD= half a right angle; and since D is a right angle, DAO is also = half a right angle.

Therefore, $DO=AD=x$, and $AO=\sqrt{2x^2}=x\sqrt{2}$; and, consequently, $CO=x\sqrt{2}+d$, and $CD=x+x\sqrt{2}+d=(1+\sqrt{2})x+d$.

$$\text{Now, } AD^2+DC^2=AC^2;$$

$$\text{or } x^2+\{(1+\sqrt{2})x+d\}^2=h^2,$$

$$\text{or } \{1+(1+\sqrt{2})^2\}x^2+2d(1+\sqrt{2})x=h^2-d^2,$$

$$\text{or } (4+2\sqrt{2})x^2+2d(1+\sqrt{2})x=h^2-d^2,$$

$$\text{or } x^2+2d\frac{1+\sqrt{2}}{4+2\sqrt{2}}x=\frac{h^2-d^2}{4+2\sqrt{2}}.$$

Whence, by the solution of this equation, we have

$$x=-\frac{1+\sqrt{2}}{4+2\sqrt{2}}d\pm\sqrt{\left\{\left(\frac{1+\sqrt{2}}{4+2\sqrt{2}}\right)^2d^2+\frac{h^2-d^2}{4+2\sqrt{2}}\right\}}$$

Where the proper numeral values of h and d being substituted, those of x, $x\sqrt{2}=AO$, and of $d+x\sqrt{2}=OC$, will become known.

Let these be denoted by m and n; that is, $AO=m$, and $OC=n$;

Then, in the right angled triangle ABC, we have given the hypothenuse $AC=h$, and the lengths of two lines OC and OA, drawn from the acute angles to the centre of the inscribed circle, to find the two sides AB, BC.

Now, let fall the three perpendiculars, OE, OF, OG, and make $AG=y$, and $GC=z$; then, follows, from (Euc. I. 47,) that

$$z^2-y^2=n^2-m^2,$$
$$\text{also, } z+y=h.$$

$$\text{Whence, by division, } z-y=\frac{n^2-m^2}{h},$$

$$\text{Consequently, } z=\frac{h^2+n^2-m^2}{2h}$$

$$y=\frac{h^2-n^2+m^2}{2h}$$

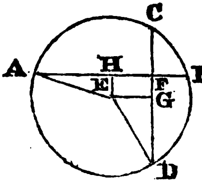
Where both segments become known, and consequently also

$$OG=\sqrt{(n^2-z^2)},$$

$$\text{or } OG=\sqrt{\left\{n^2-\frac{(h^2+n^2-m^2)^2}{4h^2}\right\}}$$

Whence AG, GC, and OG being known, we have also $AB=AG+OG=5.87447$, and $BC=GC+OG=8.08004$.

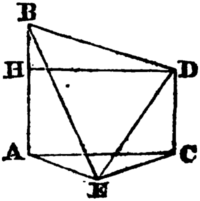
Ex. 19. Let ABC be the required circle, $AB=a$, and $DC=b$, the two given lines, and $EF=c$, the given distance.



Also, draw the two perpendiculars EH, EG; and join AE, ED. Let AE, or $ED=x$; then, $EH^2=x^2-\frac{1}{4}a^2$, and $EG^2=HF^2=x^2-\frac{1}{4}b^2$; but $EH^2+HF^2=EF^2=c^2$; therefore, $x^2-\frac{1}{4}a^2+x^2-\frac{1}{4}b^2=c^2$, and, consequently, $x=\sqrt{(\frac{1}{8}a^2+\frac{1}{8}b^2+\frac{1}{2}c^2)}$. Whence, the diameter is given $=\sqrt{(\frac{1}{2}a^2+\frac{1}{2}b^2+2c^2)}$, the Answer.

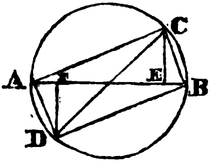
NOTE.—The answer in the Introduction is erroneous.

Ex. 20. This question, of which the figure is as follows, does not require the assistance of algebra.



For $BD=\sqrt{(HB^2+HD^2)}=\sqrt{(400+14400)}=\sqrt{14800}$,
Whence, ED and BE are each $=\sqrt{14800}$;
Consequently, $EC=\sqrt{(ED^2-DC^2)}=\sqrt{(14800-6400)}=\sqrt{8400}=20\sqrt{21}$; and $AE=\sqrt{(BE^2-AB^2)}=\sqrt{(14800-10000)}=\sqrt{4800}=40\sqrt{3}$, as required.

Ex. 21. Let ACBD be the given trapezium,
Where $AD=6$, $DB=4$, $CB=5$, and $CA=3$.



Then, draw the diagonal AB, and let fall upon it the two perpendiculars CE and DF, and make $CE=p$, and $DF=p'$; also, put the required diameter $=x$. Then, (by Euc. VI. C.)

$$p\,x=5\times3=15$$
$$p'x=4\times6=24,$$
$$\text{or } x(p+p')=39.$$

In the same manner, calling q and q' two perpendiculars falling on the diagonal CD, we have

$$q\,x=3\times6=18,$$
$$q'x=4\times5=20,$$
$$\text{or } x(q+q')=38.$$
$$\text{Whence, } (p+p') : (q+q') :: 39 : 38.$$

But the sum of these perpendiculars into double their respective diagonals are equal to the area of the trapezium; whence, AB : CD :: 38 : 39.

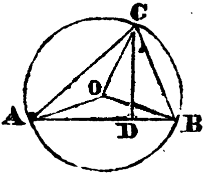
Also, (Euc. VI. D,) $AB\times CD=AC\times BD+AD\times CB$;

Let, therefore, $AB=y$, and $CD=z$;

$$\text{then, } y : z :: 38 : 39,$$
$$\text{and } yz=42;$$
$$\text{Whence, } y=\sqrt{\left(\frac{42\times38}{39}\right)}, \text{ and } z=\sqrt{\left(\frac{42\times39}{38}\right)}.$$

Now, the three sides of each of the triangles ABC and ADB being known, the perpendiculars CE and DF are readily determined. Let, therefore, these be denoted as above by p and p', and we shall have from our first equation, $x=\dfrac{39}{p+p'}$, the diameter sought.

Ex. 22. This question, of which the figure is as follows, is nothing more than having the three sides of a triangle given to find the radius of the circumscribing circle.



Let, therefore, ABC be the given triangle, of which $AC=25=a$, $AB=30=b$, $CB=20=c$, and O the required

centre. From which the perpendicular CD is readily found.

For, by putting AD$=x$, and DB$=y$, we have

$$x^2-y^2=a^2-c^2$$
$$x+y=b$$

$$\text{Whence, } x-y=\frac{a^2-c^2}{b}$$

$$\text{By addition, } x=\frac{b^2+a^2-c^2}{2b}.$$

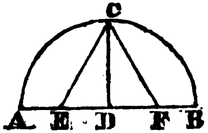
$$\text{And hence, CD}=\sqrt{\left\{a^2-\frac{(b^2+a^2-c^2)^2}{4b^2}\right\}}=\frac{25}{4}\sqrt{7}.$$

Again, (Euclid vi. c.) Diam. $\times$CD$=$AC$\times$CB,

$$\text{Whence, Diam.}=\frac{25\times20}{\frac{25}{4}\sqrt{7}}=\frac{80}{\sqrt{7}}=\frac{80}{7}\sqrt{7}=30.237116;$$

and, consequently, OA, OB, or OC$=15.118558$, the distance sought.

Ex. 28. Let ACB be the proposed semicircle, ECF the given equilateral triangle, whose area $=100=a$, and whose perpendicular CD, which is the radius of the circle, is required.



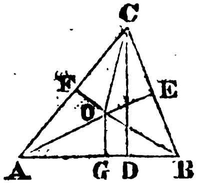
Let CE, the side of the triangle, $=x$; then, ED$=\frac{1}{2}x$, and CD$=\sqrt{(x^2-\frac{1}{4}x^2)}=\frac{1}{2}x\sqrt{3}$.

But DC$\times$ED$=\frac{1}{2}x\times\frac{1}{2}x\sqrt{3}=100$, the area.

$$\text{Whence, } \tfrac{1}{4}x^2\sqrt{3}=100, \text{ or } x=\sqrt{\frac{400}{\sqrt{3}}}=20\sqrt[4]{\tfrac{1}{3}}.$$

Consequently, $\frac{1}{2}x\sqrt{3}=10\sqrt[4]{3}=$radius,
and, therefore, $20\sqrt[4]{3}$, the diameter.

Ex. 24. Here, the rectangle of the perpendicular and diameter of the circumscribing circle of any triangle is equal to the rectangle of the two sides from which the perpendicular is drawn. (Euc. VI. C.)



Whence, the present question reduces to this, i. e. given the perpendicular $=p$, the radius of the inscribed circle OG, OE, or OF$=r$, and the product of the two sides $=2\text{R}p$, to find the sides.

Let, therefore, the segment $\text{AD}=z+x$, and $\text{DB}=z-x$.

Then, (Euc. I. 47,) $\text{AC}=\sqrt{\{(z+x)^2+p^2\}}$

$$\text{BC}=\sqrt{\{(z-x)^2+p^2\}}$$

$$\text{AB}=2z.$$

Also, because $\text{AB}\times\text{CD}=(\text{AB}+\text{BC}+\text{AC})\times\text{OG}$, we have

$$\sqrt{\{(z+x)^2+p^2\}}\times\sqrt{\{(z-x)^2+p^2\}}=2\text{R}p, \text{ and}$$

$$[\sqrt{\{(z+x)^2+p^2\}}+\sqrt{\{(z-x)^2+p^2\}}]\times r=2(p-r)z.$$

Whence, squaring the latter equation, and substituting for the double rectangle, we have

$$2z^2+2x^2+2p^2+4\text{R}p=\frac{4(p-r)^2z^2}{r^2}$$

$$\text{Or } x^2=\frac{2(p-r)^2}{r^2}z^2-z^2-p^2-2\text{R}p$$

$$\text{or, by putting } \frac{2(p-r)^2}{r^2}-1=m, \text{ and } p^2+2\text{R}p=n$$

$$x^2=mz^2-n.$$

Also, squaring the first equation, and multiplying,

$$(z^2-x^2)^2+2p^2(z^2-x^2)+p^4=4\text{R}^2p^2;$$

In which, substituting for x^2 the value found above, we have $(z^2-mz^2+n)^2+2p^2(z^2+mz^2-n)=4R^2p^2-p^4$;

That is, $z^4(1-m)^2+2n(1-m)z^2+2p^2(1+m)z^2+n^2-2p^2n=4R^2p^2-p^4$; which reduces to

$$z^4+\frac{2n(1-m)+2p^2(1+m)}{(1-m)^2}z^2=\frac{4R^2p^2-p^4-n^2+2p^2n}{(1-m)^2}$$

Where, by re-establishing the value of n, the second side of the equation becomes $=0$, and there remains

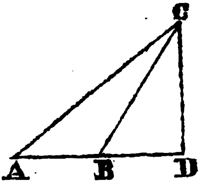
$$z^2=\frac{2n(m-1)-2p^2(m+1)}{(m-1)^2}$$

Or, by substituting for m and n their respective values, we have

$$z=\frac{r\sqrt{(2Rp-4Rr-r^2)}}{p-2r}$$

Whence, $2z=\dfrac{2r\sqrt{(2Rp-4Rr-r^2)}}{p-2r}$ the base; from which the other sides may be determined.

Ex. 25. Let ABC be the proposed triangle, in which AB$=2a$, and CD$=a$.



Then, assuming AD$=a+x$, and DB$=a-x$, or $-(x-a)$, we have AC$=\sqrt{(2a^2+2ax+x^2)}$

BC$=\sqrt{(2a^2-2ax+x^2)}$

And, by the question,

$$(2a^2+2ax+x^2)^{\frac{3}{2}}+(2a^2-2ax+x^2)^{\frac{3}{2}}=24a^3.$$

Let $2a^2+2ax+x^2=m$, and $2a^2-2ax+x^2=n$,

$$\text{Then, } m^{\frac{3}{2}}+n^{\frac{3}{2}}=24a^3$$

$$\text{And, by squaring, } m^3+n^3+2m^{\frac{3}{2}}n^{\frac{3}{2}}=576a^6,$$

$$\text{Or } 4m^3n^3=(576a^6-m^3-n^3)^2$$

Where, substituting the above values of m and n, the equation reduces to this; viz. $x=a\sqrt{\frac{2}{3}}$,

$$\text{Whence, AD}=(1+\sqrt{\tfrac{2}{3}})a, \text{ and DB}=(1-\sqrt{\tfrac{2}{3}})a.$$

Which, being negative, shows that the perpendicular falls on the base produced.

$$\text{Therefore, AC}=a\sqrt{(\frac{24}{3}+\frac{4}{3}\sqrt{6})}$$

$$\text{And BC}=a\sqrt{(\frac{24}{3}-\frac{4}{3}\sqrt{6})}$$

And this, by extracting the roots, gives

$$\text{AC}=a(2+\tfrac{1}{3}\sqrt{6})$$
$$\text{BC}=a(2-\tfrac{1}{3}\sqrt{6})$$

Which are the two sides required.

SOLUTION TO EX. 24, PAGE 177.

Let x^2-1, and y^2-1, be the two numbers sought; then each of them, when increased by unity, will be a square; we have, therefore, only to make $x^2+y^2-1=\square=v^2$, and $x^2-y^2+1=\square=u^2$. From the first we get $y^2=v^2-x^2+1$, and by adding the first and second, $2x^2=v^2+u^2$. This last condition will be satisfied by making $x=(r^2+s^2)n$, $v=(2rs-r^2+s^2)n$, and $u=(2rs+r^2-s^2)n$. Then $y^2=v^2-x^2-1$ will be $=4n^2rs(r^2-s^2)+1=\square$. Assume its side $=2nm-1$; then $4n^2rs(r^2-s^2)+1=4n^2m^2-4nm+1$, and $nrs(r^2-s^2)=nm^2-m$; hence $n=\dfrac{m}{m^2-rs(r^2-s^2)}$ where m, r, s, may be taken at pleasure.

If $r=4$, and $s=3$, then $n=\dfrac{m}{m^2-84}=-3$, (if m be taken $=9$,) hence $x=n(r^2+s^2)=-75$, $y=2nm-1=-55$, $x^2-1=5624$, and $y^2-1=3024$, the answer in the book.